# Tautological classes on the moduli space of $K3$ surfaces

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We denote by $\mathcal{K}_\ell$ the moduli stack of quasipolarized $K3$ surfaces $(X, H)$ of degree $H^2 = 2\ell$, and let

$$\pi : \mathcal{X} \to \mathcal{K}_\ell$$

be the universal surface, equipped with the universal quasipolarization $\mathcal{H} \to \mathcal{X}$.

The Hodge line bundle

$$V = \left(R^2\pi_\star \mathcal{O}_{\mathcal{X}}\right)^{-1}$$

gives rise to a natural divisor class

$$\lambda = c_1(V),$$

generating a subring $\langle\lambda\rangle \subset A^\star(\mathcal{K}_\ell)$ in the Chow of $\mathcal{K}_\ell$. In [GK], the authors consider the Chern classes $c_1 = \pi^\star\lambda$ and $c_2$ of the relative cotangent bundle $\Omega^1_{\mathcal{X}/\mathcal{K}_\ell}$, and calculate that

$$\pi_\star\, c_2^m \in \langle\lambda\rangle, \text{ for all } m.$$

Beyond the universal surface $\mathcal{X}$, we may contemplate more general geometric structures over $\mathcal{K}_\ell$, and could ask: do they give rise to new natural classes in the Chow ring of $\mathcal{K}_\ell$ or does the $\lambda$-ring entirely capture the tautological cycle structure of $\mathcal{K}_\ell$?

As a concrete example, for a fixed integer $n$, we consider the relative Hilbert scheme of $n$ points

$$\pi : \mathcal{X}^{[n]} \to \mathcal{K}_\ell.$$

(For simplicity we let $\pi$ denote the projection to $\mathcal{K}_\ell$ in all considered contexts.) We let $\mathbb{D} \subset \mathcal{X}^{[n]}$ be the natural diagonal divisor of subschemes whose support points are not all distinct. In other words, fiberwise over a quasipolarized $(X, H)$, $\mathbb{D}$ consists of the length $n$ zero-dimensional subschemes of $X$ supported at at most $n-1$ distinct points of $X$. We let $\delta \in A^1(\mathcal{X}^{[n]})$ be the corresponding Chow class, and ask

**Question 1.** *Are the pushforwards $\pi_\star\,\delta^m$ for $m > 2n$ in the $\lambda$-ring?*

The Hilbert scheme can be viewed as the relative moduli stack of rank 1 torsion free sheaves of trivial determinant and second Chern number $-n$. More generally, it is natural to consider spaces of higher rank sheaves on a $K3$, as the surface varies in moduli. We restrict attention to the open substack $\mathcal{K}^\circ_\ell \subset \mathcal{K}_\ell$ where the line bundle $\mathcal{H}$ over the universal surface is ample, and construct

$$M[v] \to \mathcal{K}^\circ_\ell,$$

the moduli space of $\mathcal{H}$-semistable sheaves with rank $r$, determinant $d\mathcal{H}$ and Euler characteristic $a-r$ over the fibers of $\pi : \mathcal{X}^\circ \to \mathcal{K}_\ell^\circ$. Over a fixed polarized $K3$ surface $(X, H)$, the moduli space consists of sheaves $F$ with Mukai vector

$$v(F) = \mathrm{ch} F \sqrt{\mathrm{todd}\, X} = r + dH + a[\mathrm{pt}] \,\in\, H^\star(X, \mathbb{Z}).$$

We may consider an additional Mukai vector $w = s + eH + b[\mathrm{pt}] \in H^\star(X, \mathbb{Z})$, complementary in the sense that

$$\chi(v \cdot w) = 0 \;\text{ on }\; X.$$

We form the second relative moduli space $M[w] \to \mathcal{K}_\ell^\circ$, and note that the product

$$\pi : M[v] \times_{\mathcal{K}_\ell^\circ} M[w] \;\to\; \mathcal{K}_\ell^\circ$$

comes endowed with a canonical Brill-Noether locus

$$\{(X, H, E,\, F) \;\text{ so that }\; \mathbb{H}^0(X, E \otimes^{\mathbf{L}} F) \neq 0\} \subset M[v] \times_{\mathcal{K}_\ell^\circ} M[w], \tag{1}$$

which is expected divisorial. The corresponding line bundle $\Theta \to M[v] \times_{\mathcal{K}_\ell^\circ} M[w]$ is in any case always defined. We ask

**Question 2.** *Is the Chern character $ch(\mathbf{R}\pi_\star \Theta)$ in the ring generated by the Hodge class $\lambda = -c_1(R^2\pi_\star \mathcal{O}_{\mathcal{X}^\circ})$ ?*

## References

[GK] G. van der Geer, T. Katsura, *Note on tautological classes of moduli of $K3$ surfaces*, Mosc. Math. J. 5 (2005), 775 – 779.