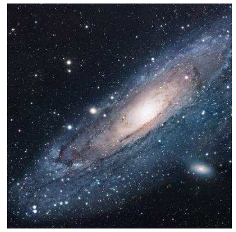
SINGULARITIES **Math118, O. Knill**

ABSTRACT. Singularities for the n-body problem can occur when bodies collide or when bodies escape to infinity. A theorem of van Zeipel shows that these are the two only possibilities.

OPEN PROBLEM. Lets start with a major open problem in celestial mechanics.

> Is it true that the Newtonian n-body problem has a full $3d$-dimensional Lebesgue measure set of initial conditions, for which the solutions exist for all times? In other words, can the singularity set have positive measure?



COLLISIONS.

If $x(t) \to \Delta$ for $t \to \tau$, then $x(\tau)$ is called a **collision singularity**. Collisions can already occur in the 2-body problem, if the total angular momentum of the two bodies is zero. Analysing collision singularities involving more than two bodies helps to understand what happens when particles move close to such collision configurations. It is known that initial conditions leading to collisions are rare in the n-body problem. Noncollision singularities in which particles escape to infinity in finite time exist already for the 5-body problem.

Our galaxy and M31, the Andromeda galaxy, form a relatively isolated system known as the **local group**. The center of mass of M31 approaches the center of mass our galaxy with a velocity of 119 km/s. In about 10^{10} years, these galaxies are likely to collide. Such a collision would have dramatic consequences for both systems. Nevertheless, even a direct encounter would probably not lead to any collision of stars.

EXISTENCE OF SOLUTIONS.

> For every point (x, y) in phase space, there exists $\tau = \tau(x, y)$ such that for $t \in [0, \tau(x)]$ the Newtonian n-body equations have a unique solution (x^t, y^t). Moreover, if K is a closed and bounded subset in the phase space, then there exists $\delta > 0$ such that (x^t, y^t) is outside K for $t \in [\tau - \delta, \tau]$.

(i) The first statement follows from a general existence theorem for differential equation $\dot{x} = f(x)$ on a subset M of Euclidean space. The function $\dot{x} = f(x)$ is Lipshitz continous on a bounded open set in M.

(ii) For any compact (closed and bounded) set K, there is a time $\tau_K = \min_{x \in K} \tau(x) > 0$ such that for all initial conditions $x \in K$, a solution exists in the time interval $[0, \tau_K)$.
Therefore, if $x(t)$ exists in the interval $[0, \tau)$ and the solution can not be extended beyond τ, then for $t \in (\tau - \tau_K, \tau]$, $x(t)$ is outside K.

SINGULARITIES. A point $(x, y) \in (T^*M)^n$ is called a **singularity** if $\tau(x, y) < \infty$. A singularity is called a **collision** if there exists $x \in \Delta$ such that $x^t \to x$. A singularity which is not a collision is a **pseudo collision** or a **non collision singularity**.

The existence theorem shows that if a singularity is approached, then the some velocities become unbounded. It is not possible that posititions become unbounded but velocities stay bounded.

PAINLEVE THEOREM. If (x, y) is a singularity, then $|U(x^t)| \to \infty$ for $t \to \tau(x, y)$. In other words, the minimal distance between two particles goes to zero. This result holds in any dimensions and for any potential $U = u(|x|)$ satisfying $u(r) \to \infty$ for $r \to 0$ and such that $u \in C^2([\epsilon, \infty))$ for every $\epsilon > 0$.

PROOF. Assume the contrary: there exists $\delta > 0$ such that $\min_{i \neq j} |x_i^t - x_j^t| \geq \delta$ for $t \in [0, \tau)$. We want to show that τ is not maximal.

(i) The differential equation $\dot{x} = f(x)$ with $|f| \leq M$ in $B_r(x_0)$ and $f \in C^1$ has a solution x^t with $x^0 = x_0$, as long as $|t| \leq r/M$. The piece of orbit $\{x^t\}_{t \in [0, r/M]}$ is contained in $B_r(x_0)$.

Proof. See the proof of the Cauchy-Piccard existence theorem.

(ii) There exists M such that $|\nabla_x U| \leq M$ for $x \notin B_r(x^0)$.

Proof. We have $0 \leq -U \leq C/\rho$, where C is a constant depending only on n and the masses m_j. Therefore, we have $|\nabla_x U| \leq C/\rho^2$.

(iii) There exists M such that $|y_j| \leq M$.

Proof. This follows from the decomposition of the energy $H = K + U$ and the boundedness of U. $\sum_{j=1}^d y_j^2/2m_j \leq H + 2M^2/d$.

(iv) For t arbitrarily close to $\tau(x, y)$, we can extend the solution for the time interval $[0, r/2M]$.
Proof. Using $(ii), (iii)$, we can apply (i).

MOMENT OF INERTIA. The number $I(x) = \sum_{i=1}^n m_i |x_i|^2$ the **moment of inertia** of the configuration.

LAGRANGE-JACOBI FORMULA. $\frac{1}{2}\ddot{I}(x^t) = U(x^t) + 2H(x^t, y^t) = T(y^t) + H(x^t, y^t)$, where $H(x, y) = T(y) - U(x)$ is decomposition of the energy into kinetic and potential energy.

PROOF. From $\frac{1}{2}\dot{I} = \sum_{j=1}^d m_j (x_j, \dot{x}_j)$, we get

$$\begin{aligned} \frac{1}{2}\ddot{I} &= \sum_{j=1}^d m_j (x_j, \ddot{x}_j) + 2T = \sum_{j=1}^d (x_j, -\nabla_{x_j} U(x)) + 2T \\ &= U + 2T = -U + 2H = T + H \; . \end{aligned}$$

We have used that U is homogeneous of degree -1: $U(\lambda x) = \lambda^{-1} U(x)$ which gives with the Euler identity $(x, \nabla_x U) = -U$.

REMARK TO 4D. Interesting is the analoguous case in $n = 4$, where U is homogeneous of degree -2. Then $\frac{1}{2}\ddot{I} = 2H$ is constant. This shows that we have in the case of a negative initial energy $H < 0$ always collapse in finite time and that solutions can stay bounded only on the energy surface $H = 0$. You have this fact in the case of the Kepler problem in four dimension.

SUNDMAN-VAN ZEIPEL LEMMA If (x, y) is a singularity, there exists $I^* = I(x^{\tau(x,y)}) \in [0, \infty]$ such that $I(x^t) \to I^*$ for $t \to \tau(x, y)$. The same relation holds for potentials for which $x \cdot \nabla_x U(x) + U(x)$ is globally bounded.

PROOF. From the Lagrange formula and the theorem of Painlevé, we see that $\ddot{I} > 0$ for t near $\tau(x, y)$. This implies that $\dot{I}$ is monotonically increasing and one can assume that $\dot{I}$ is always positive or always negative in the interval $[t, \tau]$ because one could else, if it changes sign, make the interval smaller. The positive function I is therefore monotonic and has a limit.

VAN ZEIPEL's THEOREM. This is a heavy theorem. Even so the proof had been simplified considerably by McGehee, its not possible to hide that this is a relatively deep result:

> THEOREM. If (x, y) is a singularity, then $I(x^{\tau(x,y)}) < \infty$ if and only if (x, y) is a collision. In other words, $I(x^{\tau(x,y)}) = \infty$ if and only if (x, y) is a pseudo-collision.

The proof follows closely McGehee's 1986 paper.

PROOF (i) Clusters. Denote by ω a partition of the set $N = \{1,\dots,n\}$. For $\mu \subset N$, define $\Delta_\mu = \{x \in \mathbf{R}^{3n} \mid i,j \in \mu \Rightarrow x_i = x_j\}$ and $\Delta_\omega = \bigcap_{\eta \in \omega} \Delta_\mu$.

PROOF (ii) New scalar product. Consider the scalar product in $\mathbf{R}^{3n}$ by $< x, x' >= \sum_j m_j(x_j, x'_j)$, where $(\cdot,\cdot)$ is the standard dot product in $\mathbf{R}^3$. The norm $||x||$ of x in this scalar product allows to rewrite the moment of inertia as $I(x) = ||x||^2$.

PROOF (iii) Orthogonal decomposition. Define for $\mu \subset N$ the linear map $\mathbf{R}^{3n} \to \mathbf{R}^3$

$$x \mapsto \pi_\omega x = c_\mu x = \sum_{i\in\mu} m_i x_i / \sum_{i\in\mu} m_i$$

and the linear map π_ω from $\mathbf{R}^{3n}$ to $\mathbf{R}^{3n}$. We have $(\pi_\omega x)_i = \pi_\mu x$ if $i \in \mu$. This is an orthogonal projection with range Δ_ω and kernal $\Gamma_\omega = \{\sum_{j\in\mu} m_j x_j = 0 \,\forall \mu \in \omega\}$. Denote by $\Pi_\omega = Id - \pi_\omega$ the orthogonal projection onto Γ_ω. Write $x = \pi_\omega x + \Pi_\omega(x) = z + w$.

PROOF (iv) Moment of inertia. Define $I_\omega(x) = ||\pi_\omega x||^2 = \sum_{\mu\in\omega}(\sum_{j\in\mu} m_j)|c_\mu x|^2$. Denote by J_μ the moment of inertia of a subsystem having particles $j \in \mu$ and by $J_\omega = \sum_{\mu\in\omega} J_\mu$ the sum of these moment of inertias. The equation

$$||x||^2 = ||\pi_\omega x||^2 + ||\Pi_\omega x||^2 = I_\omega(x) + J_\omega(x)$$

means that the total moment of inertia is the sum of the moment of inertias of the subsystems and the fictious system obtained from the center of masses of the subsystems.

RROOF (v) Potential energy. Define $U_{ij}(x) = \frac{1}{2}\frac{m_i m_j}{|x_i - x_j|}$ for $i \neq j$ and $U_{ij}(x) = 0$ if $i = j$. Let $V_\mu(x) = \sum_{i,j\in\mu} U_{ij}$ be the potential energy of the subsystem μ and $V_\omega(x) = \sum_{\mu\in\omega} V_\eta(x)$ the sum of the potential energies of the subsystems of a partition ω. Define $U_{\mu\nu}(x) = \sum_{i\in\mu,j\in\nu} U_{ij}(x)$ if $\mu \cap \nu = \emptyset$ and $U_{\mu\nu}(x) = 0$ else. The potential energy due to the interaction of the subsystems is $U_\omega = \sum_{\mu,\nu\in\omega} U_{\mu\nu}$. The total potential energy $U(x)$ can be written as

$$U(x) = U_\omega(x) + V_\omega(x) .$$

PROOF (vi) Dynamics. For $z \in \Delta_\mu$, we have $V_\omega(x+z) = V_\omega(x)$ which gives $V_\omega(x + \pi_\omega y) = V_\omega(x)$ for all $y \in \mathbf{R}^{2n}$. Differentiation of this with respect to y and putting $y = 0$ gives $\nabla V_\omega(x)\pi_\omega = 0$. Because π_ω is orthogonal, we have therefore $\pi_\omega \nabla V_\omega(x) = 0$. Applying the projection π_ω on $\ddot{x} = \nabla U(x) = \nabla U_\omega + \nabla V_\omega$ gives

$$\ddot{w} = \pi_\omega \ddot{x} = \pi_\omega \nabla U_\omega ,$$

from which we derive

$$\frac{d^2}{dt^2} I_\omega(x) = \frac{d^2}{dt^2} < \pi_\omega x, \pi_\omega x >= 2 < \pi_\omega \dot{x}, \pi_\omega \dot{x} > +2 < \pi_\omega x, \pi_\omega \nabla U_\omega(x) > \ .$$

PROOF (vii) Statement of the goal: We assume that $I(x^t) \to I^* < \infty$ and show that x^t converges.

RROOF (viii) The collision set Δ^*. By assumption on the theorem, the set

$$\Delta^* = \bigcap_{t<\tau(x,y)} \overline{O(t,t^*)} \subset \Delta$$

with $O(a,b) = \{x^t\}_{t\in(a,b)}$ is nonempty and compact. For each partition ω define $\Delta^*_\omega = \Delta^* \cap \Delta_\omega$. From the partitions ω with Δ^*_ω we choose a partition with minimal cardinality and fix this partition for the rest of the proof.

PROOF (ix) Bound the force in a neighborhood G of Δ^*. Since Δ^* is compact we can find an open neighborhood G of Δ^* and a constant M such that

$$||\nabla U_\omega||, | < \pi_\omega x, \nabla U_\omega(x) > | \leq M .$$

PROOF (x) If Δ^* is a subset of Δ_ω, then x^t converges.
If $\Delta^* \subset \Delta_\omega$, then $z^t = \pi_\omega x^t$ converges for $t \in \tau(x,y)$. There exists t_2 such that $x^t \in G$ for $t \in (t_2, \tau(x,y))$. From $\ddot{w} = \pi_\omega \nabla U_\omega(x)$ and the bound in (ix), we get $||\ddot{w}|| \leq M$ for $t \in (t_2, \tau(x,y))$. It follows that w^t approaches a limit w^* for $t \to \tau(x,y)$. Hence $x^t = w^t + z^t \to w^* + 0$ converges.

PROOF (xi) The situation that Δ^* is a not a subset of Δ_ω is not possible. Assume Δ^* is a not a subset of Δ_ω. In claim (ix) below, we will derive a contradiction and so finish the proof of the theorem.

PROOF (xii) Definition of a compact set $K_\sigma \subset \mathbf{R}^{3n}$. Choose a bounded open subset B of Δ_ω such that $\Delta^*_\omega \subset B \subset \overline{B} \subset \Delta_\omega \subset G$. Let D_σ denote an open ball of radius σ in the linear space Γ_ω. Define the compact set

$$K_\sigma = \overline{B} \times \overline{D}_\sigma .$$

Since the boundary δB of B is compact and B does not intersect Δ^*, there exists σ_0 and $t_0 < \tau$ such that $O([t_0,\tau)) \cap \overline{D}_{\sigma_0} \times \delta B = \emptyset$. We can choose σ_0 so small that additionally $K_{\sigma_0} \subset G$.
Since by our assumption, Δ^* is not a subset of Δ_ω, there exists $0 < \sigma < \sigma_0$ such that for infinitely many values of t close to t^*, we have $x^t \notin K_\sigma$. Choose and fix σ with this property.

PROOF (xiii) Definition of a time t_1. Chose t_1 so small that

$$|I(x^t) - I^*| \leq \frac{\sigma^2}{12}, \ \forall t_1 \leq t < t^* .$$

PROOF (xiv) Definition of a time interval $I = [a,b]$ with some properties. There exists an interval $I = [a,b]$ such that

1) $O(I) \subset K_\sigma$ 2) $||\Pi_\omega x^a|| = ||\Pi_\omega x^b|| = \sigma^2$
3) $\min_{t\in[a,b]} ||\Pi_\omega x^t|| < \sigma^2/2$ 4) $b - a < \sigma/\sqrt{3M}$.

Proof. Because x^t comes arbitrarily often arbitrarily close to Δ^*_ω, x^t must enter and leave K_σ infinitely many often. 1) is therefore no problem for intervals arbitrarily close to τ. 3) can be met for intervals arbitrarily close to τ because x^t comes arbitrarily often arbitrarily close to Δ^*_ω, where $||\Pi_\omega x^t|| = 0$. 2) is clear because if x^t enters K_σ it can not enter through $\overline{D}_\sigma \times \delta B$ and must therefore enter through $\delta\overline{D}_\sigma \times B$.

PROOF (xv) Let $s \in (a,b)$ be such that $I_\omega(x^t)$ is maximal. Remember that $I(x^t) = I_\omega(x^t) + J_\omega(x^t)$ converges for $t \to \tau$ so that a maximum exists from (vii) and (vi).

PROOF (xvi) Derive a contradiction. From $\min_{t\in[a,b]} ||\Pi_\omega x^t|| < \frac{\sigma^2}{2}$ and $|I(x^t) - I^*| < \sigma^2/12, t_1 \leq t < t^*$ we obtain

$$I_\omega(x^s) > I^* - \frac{7\sigma^2}{12} .$$

From $||\Pi_\omega x^a|| = ||\Pi_\omega x^b|| = \sigma^2$ and $|I(x^t) - I^*| < \sigma^2/12, t_1 \leq t < t^*$ we obtain

$$I_\omega(x^b) < I^* - \frac{11\sigma^2}{12}$$

so that

$$I_\omega(x^s) - I_\omega(x^b) > \frac{\sigma^2}{3} .$$

On the other hand we have for $t \in [a,b]$

$$\frac{d^2}{dt^2} I_\omega(x) = 2 < \pi_\omega \dot{x}, \pi_\omega \dot{x} > +2 < \pi_\omega x, \pi_\omega \nabla U_\omega(x) > \geq 2 < \pi_\omega x, \pi_\omega \nabla U_\omega(x) > \geq 2M \ .$$

Because s is a local maxium of $I_\omega(x^t)$, we know that

$$I_\omega(x^s) - I_\omega(x^b) \leq M(b-s)^2 < \frac{\sigma^2}{3} ,$$

where the last inequality uses 4) from (ivx).

REMARK. Edvard Hugo von Zeipel (1873-1959) was a Swedish astronomer. Van Zeipel's result holds for every potential $U(x)$ for which one can prove a Sundman-Van Zeipel lemma. For the Newton potential in four dimensions, where $\ddot{I} = const$, we know trivially that I^* exists in $(0,\infty)$. It follows that for the graviatational Newton potential in four dimensions, there are only collision singularities. For negative energy they have full measure.