NUMBERS AND DYNAMICAL SYSTEMS — Math118, O. Knill

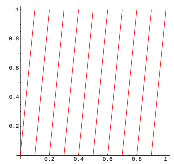
ABSTRACT. Numbers can be represented in various ways. In many cases, the representation of real numbers can be seen as a construction in symbolic dynamics.

REPRESENTATIONS OF REAL NUMBERS.

Given a finite generating partition $A_0, A_1, ..., A_n$ of the interval $[0, 1)$, define $f(y) = i$ if $y \in A_i$ and a map $T : [0, 1) \to [0, 1)$ we can look at the orbit of a point y and define the sequence $x_n = f(T^n y)$.
We are interested in cases, where the sequence x_n determines x for all x. If T is a piecewise smooth expanding map, then this is the case.

> Many representations of numbers as sequences of a finite symbols is described by symbolic dynamics.

DECIMAL EXPANSION. Let $T(x) = 10x$ and $f(x) = [10x]$ where $[r]$ is the **integer part** of r. Let $A_0, A_1, ...A_9$ be the intervals defined by $A_k = \{f(x) = k\}$.
This is the decimal expansion of x. From the sequence a_j, we can reconstruct $x = \sum_{j=1}^{\infty} a_j 10^{-j}$.



CONTINUED FRACTION EXPANSION. Take $T(y) = 1/y$ mod 1 and $f(y) = [1/y]$. For a point y, define the sequence $a_n = f(T^n(y))$. It is called the **continued fraction expansion** of y. if y is a rational number, then

$$y = [a_0; a_1, ...a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + ... \cfrac{1}{1 + a_n}}} = p_n/q_n$$

If y is an irrational number, then

$$y = [a_0; a_1, ...a_n, ...] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + ... \cfrac{1}{1 + a_n} + ...}}$$

EXAMPLES. $\sqrt{2} = [1; 2, 2, 2, 2, 2, ...]$. Since $1/(2 + x) = x$ has the solution $\sqrt{2} - 1$.
$(\sqrt{5} - 1)/2 = [1; 1, 1, 1, 1, 1, ...]$. Since $1/(1 + x) = x$ has the solution $(\sqrt{5} - 1)/2$.
$5/7 = [0; 1, 2, 2]$

PARTIAL QUOTIENTS. The **partial quotients** p_n/q_n satisfy the recursion $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ with the initial conditions $p_{-1} = 1, p_0 = a_0, q_{-1} = 0, q_0 = 1$ so that $p_0/q_0 = a_0, a_1/b_1 = a_0 + 1/a_1 = (a_0 a_1 + 1)/a_1$.

CONVERGENCE ESTIMATES. One can write the second order recursion as a first order recursion $\begin{bmatrix} p_n \\ p_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ p_{n-2} \end{bmatrix}$. In the product of matrices $A^n = A_n ... A_0 = \begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix}$ each matrix $A_k = \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}$ has determinant (-1). The product has therefore the determinant $(-1)^n$. This gives the important identity

$$p_{n-1} q_n - p_n q_{n-1} = (-1)^n$$

which implies $p_{n-1}/q_{n-1} - p_n/q_n = (-1)^n/(q_n q_{n-1})$. Since $q_n \geq q_{n-1} = 1$, we have $q_n \geq n$ and $|p_{n-1}/q_{n-1} - p_n/q_n| \leq (-1)^n/n^2$ so that p_n/q_n is a Cauchy sequence. Because p_n/q_n is alternatively below and above x (look at the images of the basis vectors of A_k), we have even the bound

$$|x - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$$

SOLVING LINEAR EQUATIONS. Given a, b, c, how do we solve $ax + by = c$ for integers x, y?
Solution: we can solve $p_{n-1} q_n - p_n q_{n-1} = (-1)^n$ by making the continued fraction expansion of p_n/q_n then multiply the result with $(-1)^n c$.

EXPANSION OF PI. To find the continued fraction expansion of $x = \pi$: $\pi = 3 + 1/(7 + ...)$, look at the orbit of $x = 0.141592653...$ under the map $T(x) = \{1/x\}$ and see in which intervals they fall.

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T[x_] := Mod[1/x, 1]; S = NestList[T, Pi − 3, 10]; f[x_] := Floor[1/x]; Map[f, S]
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Mathematica has already built in the continued fraction expansion as a basic function:

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ContinuedFraction[Pi, 10]
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The result is $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, ...]$. Continued fraction expansion of π has been computed up to 10^8 terms. One can use partial quotients like

$$\pi \sim [3; 7] = 22/7 = 3.14286$$

$$\pi \sim [3; 7, 15] = 333/106 = 3.14151$$

$$\pi \sim [3; 7, 15, 1] = 355/113 = 3.14159$$

to approximate π with rational numbers. Mathematica has the reconstruction of a number from the continued fraction built in too:

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FromContinuedFraction[3, 2, 1]
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KHINCHIN CONSTANT. If $[a_0; a_1, a_2, ...]$ is the continued fraction expansion of a number, then the limit $(a_1 a_2 a_3 ... a_n)^{1/n}$ exists for almost all irrational numbers. The limit is called **Khinchins constant**. Numerical experiments indicate that this limit is obtained for π but one does not know.

β-EXPANSION. A generalization of the decimal or expansion with respect to an integer base is the **beta expansion**. For any given real number $\beta > 1$, define the map $T(x) = \beta x$ and $f(x) = [\beta x]$. One has still $x = \sum_{i=1}^{\infty} a_i \beta^{-i}$ however, the transformation is no more so easy to understand as in the integer case. For example, T_β does not preserve the length measure dx any more in general.

PERIODIC POINTS. As in any dynamical system, also for dynamical systems which define number, periodic points are important. Examples:

- **Rational points** are eventually periodic points of the decimal expansion.
- quadratic irrationals are eventually periodic points of the continued fraction expansion.
- Numbers which lead to eventually periodic orbits of the β-expansion are called **beta numbers**.

The determination whether an orbit is eventually periodic or not is nontrivial. For example it is unknown whether $\pi + e$ is rational. In other words, one does not know whether the shift on $X_{\pi+e,10}$ is eventually periodic.

BETA NUMBERS. An interesting question is for which real numbers β and $x = 1$, the attractor is a periodic orbit. If this is the case, then β is called a **beta number**. Examples are **Pisot numbers**, algebraic integers $\beta > 1$ for which all conjugates β^σ have norm $|\beta^\sigma < 1$ besides the identity. The positive root of $x^3 - x - 1 = 0$ is known to be the smallest Pisot number. If $|\beta^\sigma| \leq 1$ for any embedding and β is not a Pisot number, it is called a **Salem number**.

NORMALITY. If every word of length k in the decimal expansion of π appears with probability 10^{-k}, then π is **normal**. One does not know whether this is true. **Normality** results are hard to get. And normality with respect to one base does not mean normality with respect to an other base. Normality is a statement with respect a specific shift invariant measure and If a number is normal with respect to all bases is called **absolutely normal**. A well studied open problem is

> Is π normal with respect to any base or even absolutely normal?