SYMBOLIC DYNAMICS — **Math118, O. Knill**

ABSTRACT. We have seen shifts as cellular automata, in a horse-shoes or in Julia set. We look at this dynamical system a bit closer.

THE SHIFT. Given a finite alphabet A, define $X = A^N$ and $\sigma(x)_n = x_{n+1}$. This dynamical system is called the **one sided shift**. The shift on A^Z is called the **two sided shift**. While the later is invertible, the first is not.

SUBSHIFTS. The shift restricted to a closed shift-invariant subset X of A^Z is called a **subshift**.

EXAMPLE. Let $T(x) = x + \alpha$ mod 1 and $Y = [0, a)$ and interval. Look at all sequences obtained by taking a point x and defining $x_n = 1_Y(x + n\alpha)$, where $1_Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$. That is

$$x_n = \begin{cases} 1 & (x_0 + n\alpha) \bmod 1 \in Y \\ 0 & (x_0 + n\alpha) \bmod 1 \notin Y \end{cases} .$$

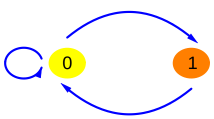
Lets assume for example, $Y = [0, 1/2)$ and $\alpha = \sqrt{2}$. With the starting point $x = 0$, we obtain the sequence $\{x_0, x_1, x_2, ...\} = \{1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, ...\}$. The image X of the map S is a closed subset of the sequences. Every orbit of the shift σ in X is dense.

A particular interesting case is $\alpha = (\sqrt{5} - 1)/2$ and $F = [0, \alpha)$. If $x_0 = 1$, then $x_1 = 0$. If $x_0 = 0$, then $x_1 = 1, x_2 = 0$. One can obtain the sequence also by applying the substitution rule $1 \to 0, 0 \to 10$. A sequence obtained like this is called a **Fibonacci sequence**. Here is part of the sequence: $..., 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0,$

EXAMPLE: SUBSHIFTS OF FINITE TYPE. Given a finite set of words K over an alphabet A. The set X of all sequences, in which the words of K do **not** appear, is called a **subshift of finite type**.
The **language** of X is the set of all words which occur in sequences of X. There is a finite set of words which can build up any sequence $x \in X$ and such that the forbidden words determine which words can be adjacent and which not. We can define a **directed graph** (V, E), which has as vertices these words and where an arrow goes from one word to an other if these words can be glued together. One says that the graph represents the subshift.

EXAMPLE. Assume $K = \{00, 111\}$ are the forbidden words, then a sequence can be ...0101101011010110101010111011.... We can get any sequence by gluing together words $w_1 = 01, w_2 = 11$ and $w_3 = 10$. The combinations $w_1 \to w_1, w_1 \to w_2, w_2 \to w_1, w_3 \to w_1, w_3 \to w_2, w_3 \to w_3$ are possible.

EXAMPLE. Let $K = \{11\}$, then X consists of all sequences, where no double 11 occur. The language of X is $\{0, 1, 00, 10, 01, 10, 000, 001, 010, 100, 101, 0000, 0001, ...\}$. With the set $V = \{00, 01, 10\}$ of words one can build any sequence. The gluing $00 \to 01, 01 \to 01, 00 \to 00$. $10 \to 10, 10 \to 01$ are possible, while the gluing $01 \to 10$ is not possible.



SOFIC SHIFTS. If X is a subshift of finite type and T is a cellular automaton map, then $T(X)$ is called a **sofic shift**.

Sophic shifts produce **regular languages**, languages accepted by finite state automata, but they are in general no more of finite type. The next example shows this.

EXAMPLE. The **even shift** is the set of all $x \in \{0, 1\}^Z$ so that between any two 1, there is an even number of 0's. The even shift is not a subshift of finite type, but it is a sofic shift. Start with the subshift of finite type, with the forbidden word 00. Take the elementary CA which gives only 0 for $1, 0, 1$ and $0, 1, 0$ and $0, 1, 1$. For example, $x = ...011110101111011011101111110111111111...$ is mapped to $y = ...0000111001001100....$ The image of this cellular automaton consists of all sequences for which 0 occurs only in blocks containing an even number.

IRREDUCIBLE SHIFTS. A subshift is called **irreducible** if the language $B(X)$ has the property if v, w are words in $B(X)$, then there is a word u in $B(X)$ such that vuw is also in $B(X)$.

PROPOSITION. A subshift (X, T) is irreducible if and only if it is transitive.

PROOF. Assume the subshift is irreducible. We show that for every n, there is an orbit which comes $1/n$ close to any point in X. To do so, make a list of all words $w_0, ..., w_L$ of length $2n + 1$ which appear in X. By assumption we can fill in words $v_1, ..., v_L$ such that $w_0 v_1 w_1 v_1 .... v_L w_L$ is part of a sequence $x \in X$. Now, $T^n(x)$ comes $1/n$ close to any point in X.
On the other hand, if (X, T) is transitive, there is a point x such that $T^n(x)$ is dense. Given two words u, w which in the language of X, there exists n such that $(T^n(x)_1 .... T^n(x)_k) = u$ and m such that $(T^m(x)_1 .... T^m(x)_l) = w$. The word v between u and w in the sequence x is the one we need to prove irreducibility.

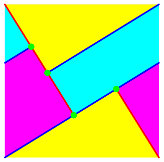
OVERVIEW. class of all subshifts $\supset$ class of all sofic shifts $\supset$ class of all shifts of finite type
CA leave the class of sofic shifts invariant because the composition of two CA is again a cellular automaton.

MINIMAL SHIFTS. A subshift (X, σ) is called **minimal**, if every orbit is dense. Note that minimal shifts can not have periodic points unless it is periodic itself.

EXAMPLE: STURMEAN SEQUENCES. **Sturmean sequences** $x_n = 1_A(t + n\alpha)$, where α is irrational and A is an interval on the circle are minimal because the irrational rotation on the circle is minimal and the symbolic map S is continuous and invertible. Because every orbit $T^n(x)$ of the irrational rotation is dense, also the corresponding orbit $S(T^n(x))$ is dense.

EXAMPLE: The full shift as well as subshifts of finite type are **not** minimal. They have many periodic orbits.

SYMBOLIC DYNAMICS. The basic construction of symbolic dynamics for a given dynamical system (Y, T) is to find a **partition** of the set X into subsets $A_0, ..., A_{n-1}$. Every point x is then assigned a sequence where $x_k = a$ if $T^k(x) \in A_a$. This **generating partition** defines a map S from Y to $X = \{0, ..., n-1\}^N$ if T is not invertible, or to $X = \{1, ..., n\}^Z$ if T is invertible. The map S conjugates (Y, T) to the subshift $(\overline{S(Y)}, \sigma)$. The map S is continuous, but it is in general neither injective nor surjective. In the homework, you deal with a a partition in case of the cat map. It is called **Markov partition**.

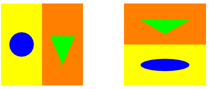


EXAMPLE. Let $T(y) = y + \alpha$ a rotation on the circle $Y = R/Z$. With $A_0 = [0, 1/2), A_1 = [1/2, 1)$, the sequence $x = S(y)$ is called a **Sturmean sequence**. The map S is a continuous map from the circle to the sequence space. But the image is not the entire space. For example, it does not contain any periodic sequences.

THE BAKER TRANSFORMATION. The baker transformation is a map on the square $Y = [0, 1) \times [0, 1)$: The map preserves area and is invertible

$$T(u, v) = \begin{cases} (2u, v/2) & , 0 \le u < 1/2 \\ (2u - 1, (v + 1)/2) & , 1/2 \le u \le 1 \end{cases} \qquad T^{-1}(u, v) = \begin{cases} (u/2, 2v) & , 0 \le v < 1/2 \\ ((u + 1)/2, 2v - 1) & , 1/2 \le v \le 1 \end{cases}$$

The inverse is obtained by switching u and v, applying T and switching u and v again. Now take the **generating partition** $A_0 = \{u \in [0, 1/2)\}, A_1 = \{u \in [1/2, 1]\}$. The symbolic dynamics of a point (u, v) defines a sequence $x \in \{0, 1\}^Z$.



THEOREM. The map S is an invertible map from the square Y to $X = S(Y)$ and $\sigma \circ S(u, v) = S \circ T(u, v)$. For any given sequence x in the image $S(Y)$, we can get back $(u, v) = S^{-1}x$ with

$$u = \sum_{k=0}^{\infty} x_k 2^{-k-1}, v = \sum_{k=-\infty}^{-1} x_k 2^k .$$

EXAMPLES:

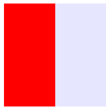
$...x_{-2}x_{-1}, x_0x_1x_2x_3...$	(u, v)
...0000, 10000..	(1/2, 0)
...0001, 00000..	(0, 1/2)
...0000, 01110..	(7/16, 0)
...0000, 11100..	(7/8, 0)
...0001, 11000..	(3/4, 1/2)
...0011, 10000..	(1/2, 3/4)
...0111, 00000..	(0, 7/8)
...1110, 00000..	(0, 7/16)

Remark: While S is injective, it is not surjective. (The point $(....0000000, 0111111...)$ is not reached, but represented by $(....0000000, 1000000...) \sim (1/2, 0)$ While the map S^{-1} is continuous, T and S are both not.

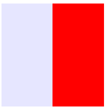
I: The binary expansion of u is $u = 0.x_0x_1x_2....$

$$u = \sum_{i=0}^{\infty} x_k 2^{-k-1} .$$

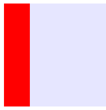
$x_0 = 0$ means that $u \in [0, 1/2)$. Note that $u = 1/2$ gives $x_0 = 1$.



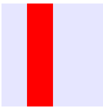
$x_0 = 1$ means that $u \in [1/2, 1)$.



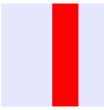
$x_0 = 0, x_1 = 0$ means $u \in [0, 1/2)$ and $2u \in [0, 1/2)$ which is equivalent to $u \in [0, 1/4)$.



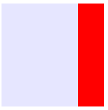
$x_0 = 0, x_1 = 1$ means $u \in [0, 1/2)$ and $2u \in [1/2, 1)$ which is equivalent to $u \in [1/4, 1/2)$.



$x_0 = 1, x_1 = 0$ means $u \in [0, 1/2)$ and $2u \in [0, 1/2)$ which is equivalent to $u \in [1/2, 3/4)$.



$x_0 = 1, x_1 = 1$ means $u \in [0, 1/2)$ and $2u \in [1/2, 1)$ which is equivalent to $u \in [3/4, 1)$.

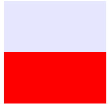


In general, fixing $x_0, ..., x_{n-1}$ determines in which of the 2^n intervals $[(k-1)/2^n, k/2^n)$ the coordinate u is.

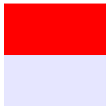
II: The binary expansion of v is $v = 0.x_{-1}x_{-2}x_{-3}....$

$$v = \sum_{k=-\infty}^{-1} x_k 2^k .$$

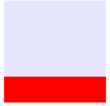
$x_{-1} = 0$ means that $v \in [0, 1/2)$. T maps the left half of the square to the lower half of the square so that T^{-1} maps the lower half of the square to the left half.



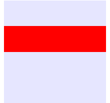
$x_{-1} = 1$ means that $v \in [1/2, 1)$.



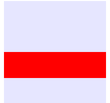
$x_{-1} = 0, x_{-2} = 0$ means $v \in [0, 1/2)$ and $2v \in [0, 1/2)$ which is equivalent to $v \in [0, 1/4)$.



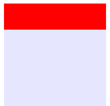
$x_{-1} = 0, x_{-2} = 1$ means $v \in [0, 1/2)$ and $2v \in [1/2, 1)$ which is equivalent to $v \in [1/4, 1/2)$.



$x_{-1} = 1, x_{-2} = 0$ means $v \in [0, 1/2)$ and $2v \in [0, 1/2)$ which is equivalent to $v \in [1/2, 3/4)$.



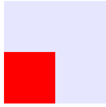
$x_{-1} = 1, x_{-2} = 1$ means $v \in [0, 1/2)$ and $2v \in [1/2, 1)$ which is equivalent to $v \in [3/4, 1)$.



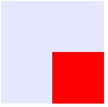
Fixing $x_{-1}, ..., x_{-n}$ determines in which of the 2^n intervals $[(k-1)/2^n, k/2^n)$ the coordinate v is.

III: Combination of part I and Part II

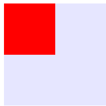
$x_{-1} = 0, x_0 = 0$ means $u \in [0, 1/2)$ and $v \in [0, 1/2)$.



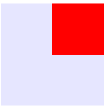
$x_{-1} = 0, x_0 = 1$ means $u \in [0, 1/2)$ and $v \in [1/2, 1)$.



$x_{-1} = 1, x_0 = 0$ means $u \in [1/2, 1)$ and $v \in [0, 1/2)$.



$x_{-1} = 1, x_0 = 1$ means $u \in [1/2, 1)$ and $v \in [1/2, 1)$.

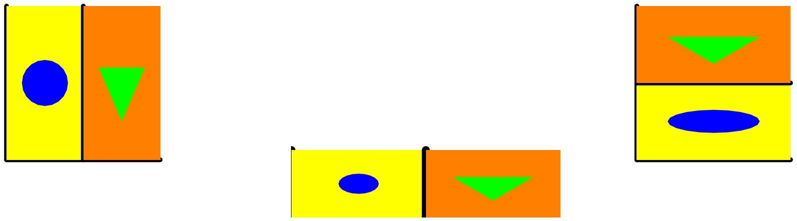


Fixing $x_{-m}, ..x_0, ..x_n$ determines in which of the 2^{n+m+1} rectangles $[(k-1)/2^n, k/2^n) \times [(l-1)/2^{m-1}, l/2^{m-1})$ the coordinate (u, v) is.

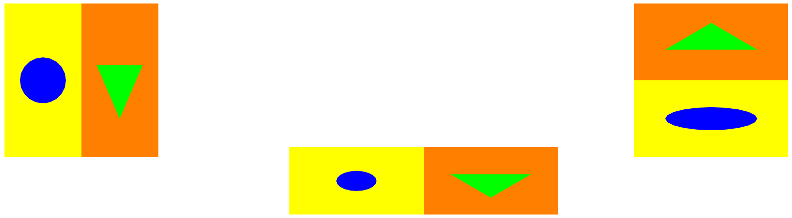
IV: Symmetry

We know $u = 0.x_0x_1x_2x_3x_4....$ Because replacing T and T^{-1} corresponds to switching u with v and replacing the partition $A_0, A-1$ with $B_0 = \{v < 1/2\}, B_1 = \{v \geq 1/2\}$, the itinery y with respect to the new partition gives $v = 0.y_0y_1y_2y_3y_4...$ Because $T(A_0) = B_0$, we have $v = 0.x_{-1}x_{-2}x_{-3}....$

BAKER MAP. In the baker map, the second rectangle is translated straight onto the first rectangle.



FAT HORSE SHOE MAP. The symbolic dynamics of the horse shoe is similar except that the second rectangle is turn around by 180 degrees. In the horse shoe, the stretching is stronger. There was a set K which never leaves the rectangle (the horse shoe is kind of a "Julia set").



TWO REMARKS. The baker map can also be conjugated to the **right shift** $\sigma x_n = x_{n-1}$. If we take the same generating partition A_0, A_1, then the formulas for S^{-1} become $u = \sum_{k=-\infty}^{0} x_n 2^{-n-1}, v = \sum_{k=1}^{n} x_n 2^{-n}$. In many treatments of symbolic dynamics of the Baker transformations, one neglects things of area zero. In that case, it does not matter, what boundary we take for the generating partition. If we want the symbolic dynamics to work for **every** point in the square Y, then we remove the right and upper boundaries in all rectangles which appear as we have done that here.