# SHIFTS IN QUADRATIC AND STANDARD MAP

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ABSTRACT. We look on this page at an analytic proof that there is an invariant shift embedded in some Hénon maps, Standard maps or quadratic maps. The proof uses the **implicit function theorem** and is based on an idea of Aubry and Abramovici called **anti-integrable limit**.

THEOREM OF DEVANEY-NITECKI. Fix $b \neq 0$. For large enough $c$, the Hénon map $H : (x, y) \mapsto (x^2 - c - by, x)$ has an invariant set $K$ such that $T$ restricted to $K$ is conjugated to the shift

$$S = (..., x_{-1}, x_0, x_1, x_2, ...) \to (..., x_0, x_1, x_2, x_3, ...)$$

on all sequences with two symbols.

PROOF. With the new parameter $a = 1/\sqrt{c}$ and the new coordinates $q = x \cdot a, p = y \cdot a$, the map becomes

$$T(q, p) \mapsto (\frac{q^2 - 1}{a} - bp, q)$$

and is equivalent to the recurrence

$$a \cdot q_{n+1} + a \cdot b \cdot q_{n-1} = q_n^2 - 1 \; .$$

We look for sequences $q_n = q(S^n x)$, where $S$ is the shift on the space of all sequence $X = \{-1, 1\}^Z$ and where $q$ is a continuous map from $X$ to $R$. We have to solve

$$a \cdot q(Sx) + a \cdot b \cdot q(S^{-1}x) - (q(x)^2 - 1) = 0 \; .$$

With the map $F : R \times C(X) \to C(X)$ defined by

$$F(a, q)(x) = a \cdot q(Sx) + a \cdot b \cdot q(S^{-1}x) - (q(x)^2 - 1)$$

this equation can be rewritten as $F(a, q) = 0$. The partial derivative $F_q(a, q)$ is

$$F_q(a, q)u = a(u(S) + b \cdot u(S^{-1})) - 2q \cdot u \; .$$

The map $F(0, q) : C(X) \to C(X)$ has the property that every function $q \in C(X)$ with values in $\{-1, 1\}$ is a solution of $F(0, q) = 0$. We take for such a solution the map $q(x) = x_0$.

The derivative $F_q(0, q)$ is the linear map

$$(F_q(0, q)u = -2q \cdot u$$

which is invertible because $q$ is bounded away from 0.

By the implicit function theorem, there exists a solution $a \mapsto q_a = G(a)$ satisfying $F(a, q_a) = 0$ for small $a$. Define $\phi_a : X \to R^2$ by

$$\phi_a(x) = (q(x), q(S^{-1}x)) \; .$$

The map $\phi_a$ is continuous, because $q$ and $T$ are continuous.
Using $F(a, q) = 0$, we check that

$$\begin{aligned} \phi_a \circ T(x) &= (q(Sx), q(x))) = (\frac{(q(x)^2 - 1)}{a} - b \cdot q(S^{-1}x), q(x)) \\ &= T(q(x), q(S^{-1}x)) = T \circ \phi_a(x) \end{aligned}$$

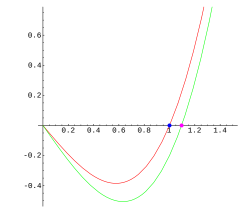
for all $x \in X$.

The map is injective because if two points $x, y$ are mapped into the same point in $R^2$ then the fact that $q_a(x)$ is near $q_0(x) = x_0$ implies $x_0 = y_0$. The conjugation $\phi_a \circ S^n(x) = T^n \circ \phi_a(x)$ gives us $T^n(x) = T^n(y)$ and so $x_n = y_n$ for all $n$.

$\phi$ has a continuous inverse because every bijective map from a compact space to a compact space has a continuous inverse. The map is indeed a homeomorphism from $X$ to a closed subset $K = \phi(X) \subset R^2$.

THE IMPLICIT FUNCTION THEOREM. Given a family $q \to F(a, q)$ of maps, parametrized by a parameter $a$. If $F(0, q_0) = 0$ and $F'(0, q_0) \neq 0$, then there exists a continuous function $q$ in some interval $I$ such that $F(a, q(a)) = 0$ for $a \in I$.

PROOF. The Newton map $T_a(q) = q - F(a, q)/F'(a, q)$ has as a stable fixed point which is the root $q(a)$. This fixed point exists for small $a$ and changes continuously with $a$.
This proof works also in infinite dimensional spaces, in which it is possible to differentiate. An example is the space $C(X)$ of continuous functions on a compact set $X$. Example: let $F(f) = f^3 + 5f$. The function $F$ maps a continuous function to a continuous unction. One has $F'(f)g = (3f^2 + 5)g$. Example: let $F(f) = f(x^2)$. Because this is a a linear map in $f$, we have $F'(f)g(x) = f(x^2)g(x)$.




HORSE SHOES IN THE STANDARD MAP. For large enough $c$, the Standard map $T : (x, y) \mapsto (2x + c\sin(x) - y, x)$ has an invariant set $K$ such that $T$ restricted to $K$ is conjugated to the shift

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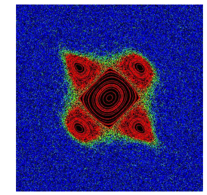
on all sequences with two symbols.



PROOF. If $T^n(q, p) = (q_n, p_n)$ is an orbit of the Standard map, then $p_n = q_{n-1}$ and so $q_{n+1} - 2q_n + q_{n-1} + c\sin(x_n) = 0$. With $\epsilon = 1/c$, this means

$$\epsilon(q_{n+1} - 2q_n + q_{n-1}) + \sin(x_n) = 0$$

Let $X$ be all $\{0, 1\}$ sequences. Consider the space of all continuous functions $q$ from $X$ to $[0, 2\pi]$.
If we find a solution $q$ to the equation

$$F(\epsilon, q) = \epsilon(q(\sigma x) - 2q(x) + q(\sigma^{-1}x)) + \sin(q(x)) = 0$$

then $q$ is a conjugation from $(X, \sigma)$ to $(q(X), T)$ showing that we can find a shift similar as the horse shoe construction does.

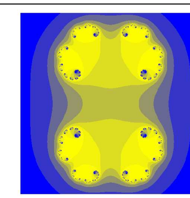
(i) There is a solution for $\epsilon = 0$: Just take $q(x) = \pi x_0$. Because $\sin(0) = \sin(\pi) = 0$, the equation $\sin(q(x)) = 0$ is satisfied.

(ii) In order to have a solution for small $\epsilon$, we compute the derivative of $L = F_q(0, q) = \cos(q)$ and see whether it is invertible. Indeed, since $L = \cos(q(x)) = \pm 1$, we can invert $L$, the inverse is actually equal to $L$. (Note that $F$ has as an argument a function $q$ and the the derivative $F_q(a, q) = \lim_{u \to q}(F(a, q + u) - F(a, q))/u$ is defined with respect to the function $q$. It was computed in the same way as derivatives with respect to real parameters.)
(iii) The implicit function theorem now assures that we can find for small $\epsilon$ a function $q_\epsilon$ which satisfies $F(\epsilon, q_\epsilon) = 0$. This function $q_\epsilon$ conjugates the shift with the standard map $T_c$ restricted to the set $K = q_\epsilon(X)$. Since $\epsilon = 1/c$, this conjugation works for large enough $c$.

JULIA SETS. The same construction works also for the map $f(z) = a(z^2 - 1)$. We look for a function $q \in C(X, C)$ such that $q(\sigma) - a(z^2 + 1) = 0$. With $\epsilon = 1/a$, this is

$$F(\epsilon, q) = \epsilon q(\sigma) - (z^2 - 1) = 0 \; .$$

For $\epsilon = 0$, the function $q(x) = (2x_0 - 1)$ is a solution. The derivative $L = F_q(0, q) = 2q$ is invertible. We have solutions for small $\epsilon$, which corresponds to large $a$. Actually, the image $q(X)$ is just the Julia set of $f$.



SUMMARY. The anti-integrable limit construction allows to get embedded shifts in a purely **analytic** way using the **implicit function theorem**. In comparison, the construction of a **horse shoe** is a **geometric** construction. Finding a **generating partition** is a more **combinatorial** task. The shift brings different areas of mathematics together.