MORE ON CA

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ABSTRACT. We add some additional remarks about CA and an open problem.

AUTOMATA ON GRAPHS. Cellular automata can be defined in any dimensions and even on any homogenous graph, where each node looks the same. A popular "two dimensional" example different from the square lattice is the hexagonal lattice. Setting up the CA story on more general graphs is nothing more than changing notation.

A general class of graphs, for which most of the theory goes over are **Cayley graphs** Γ of finitely presented groups like $G = \{a, b \mid a^2b = ba^2\}$. The graph has nodes for each word in the generators a, b and two nodes v, w are connected, if $va = w$ or $av = w$ or $w = va$ or $w = vb$.
As a metric, one first introduces the geodesic distance in the graph Γ which is the shortest number of steps (applying one of the generators of the group G) to get from one point to the other. Write $|k|$ for the distance to the origin. The distance between two configurations in $X = A^\Gamma$ is still defined as $d(x, y) = 1/(n + 1)$, where $x_k = y_k$ for $|k| \leq n$ and $x_k \neq y_k$ for some k satisfying $|k| = n$.

Hedlunds theorem still applies: a continous map on $X = A^\Gamma$ which is invariant under translations (applying the group G on the Cayley graph) is defined by a local law ϕ.

The proof we have given before applies almost word by word: the continuity of the map T forces a local law. The translational invariance and the fact that the action of the group G on the graph is transitive, implies that the law is the same at every node.

PROBLEMS WITH CA. The discretisation distroys **rotational symmetry**. In the plane, one can make CA more symmetric by using a hexagonal lattice but still, there is no rotational symmetry. Even in the limit when the cells become infinitesimally small, their stucture can be seen from the propagation of solutions.

SURJECTIVITY. Which automata are T are invertible maps on X and so homeomorphisms (every bijective map on a compact space has a continuous inverse). It is also known that an injective CA is surjective. To check injectivity, one actually can restrict to finite configurations. These results had been obtained in the 60ies. Thew fact that injectivity implies surjectivity is called a **"Garden of Eden theorem"**. The from E.F. Moore coined expression "Garden of Eden patterns" is a picturesque name for points in X, which are not in the image of T.

AN OPEN PROBLEM. An automaton T is called transitive, if it has a dense orbit in X. We have seen that the shift is transitive. We also have seen that the shift has a dense set of periodic points. F. Blanchard asks:

> Does every transitive automaton have a dense set of periodic points?

Francois Blanchard writes: "The answer, positive or negative, is a necessary step before on understands the meaning of chaos in the field. " Source: This problem can be found in Michael Misiurewicz list of open problems in dynamical systems (http://www.math.iupui.edu/ mmisiure/open)

THE SEMIGROUP OF CA. If you have a CA T and a CA S defined on the same space X, then $T \circ S$ is a new CA. So, the set of all CA is a semigroup. Historically, this was one of the original ways how CA were introducd because according to Hedlund, cellular automata are just the homomorphism on the category of subshifts. Note that the semigroup of all cellular automata is not commutative.
If you look at the set of all CA which are invertible, then the set of all these cellular automata forms a group. The identity in this group is the trivial CA, where $T(x) = x$.

A CLASS OF REVERSIBLE AUTOMATA. Given an alphabet A and an elementary automaton T defined by a function $\phi : A^3 \to A$ we can define an automaton

$$T(x, y)_i = (y_i + \phi(x_{i-1}, x_i, x_{i+1}), x_i)$$

The map T is now invertible with the inverse $T^{-1}(x, y)_i = (y_i, x_i - \phi(y_{i-1}, y_i, y_{i+1})$. It suffices to look the first coordinate because $y(t) = x(t - 1)$.

This automaton on can actually be written as an automaton on $X = B^Z$, where B is the alphabet $A \times A$. For example, for $A = \{0, 1\}$, the new alphabet B is $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$. The translation if $x_k = (0, 1)$, then this would correspond to $(x_k, y_k) = (0, 1)$ in the original picture.

CA AS MAPS ON SUBSHIFTS. If X is a **subshift** that is a shift invariant subset of A^Z, and T is a CA map, then $T(X)$ is again a subshift. It is called a factor of X. There are some properties of subshifts which stay the same after applying CA maps.

- **Topological transitive**: there exists a dense orbit.
- **Almost periodic = minimal**: every orbit is dense.
- **Uniquely ergodic**: there exists exactly one invariant measure.
- **Strictly ergodic**: minimal and uniquely ergodic.
- **Dense set of periodic orbits**: x periodic orbit: $T^n(x) = x$.
- **Prime**: Every factor of (X, T) is either trivial or isomorphic to X.
- **Totally minimal**: No factor is a finite permutation.
- **Completly positive entropy**: all non trivial factors have positive directional entropy.
- **Zero directional topological entropy**:
- **Topologically strongly mixing**: U, V open. Exists $n \in Z$ such that $U \cap T^n V \neq \emptyset$.
- **Topologically weakly mixing**: $X \times X$ is topologically transitive.
- **Uniquely ergodic, strong mixing**: $\mu(U \cap T^n V) \to \mu(U) \cdot \mu(V)$.
- **Uniquely ergodic, weakly mixing subshifts**: $X \times X$ is ergodic.
- **Sophic**: a factor of a subshift of finite type.
- **Chaotic in the sense of Devaney**: topological transitive and dense set of periodic orbits .

(If one requires additionally that the shift is not periodic, then this property is not invariant. There are shifts which have periodic factors).

> Cellular automata maps can be used to generate new subshifts with given dynamical properties!

Is this useful? It can be. If you have a complex subshift to analyze and if you can show that it is obtained by applying CA maps from a simpler shift, then you have proven that the subshift inherits the properties of the initial subshift.

ABOUT COMPLEXITY. The shift acting on all periodic sequences is not very spectacular. It just rotates a sequence. Every orbit is n periodic. Other cellular automata like rule 30 have complexer behavior when restricted to periodic sequences in the sense that there are longer periodic orbits in that space X of 2^n possible configurations. Note that T can never be transitive on X in the periodic setup because if you start with a constant sequence x, then $T(x)$ is a constant sequence. But orbits can get long.

> The complexity of a dynamical system can depend dramatically on the space, on which it is defined.

REMINDER: A linear map A like the cat map on R^2 behaves differently then the same map on the torus R^2/Z^2. The map on the torus is complex. However, when restricting the map on the set of rational points $(x, y) \in X$, the map is not complex at all: every orbit is eventually periodic.

REMINDER: The free motion of a particle in the plane is trivial. But when confined to a finite region (a billiard table), the motion can become complex. Then again, restricting this complex motion to some subset can be completely understandable like restrictiion to the invariant curve on which the dynamics is just a translation.

Talking about the complexity of a map or differential equation does not make sense per se. The set X on which one wants to understand the system is important. Complexity is often mentioned in discussions about CA. Like other **buz words**, the word is loaded with many different meanings. One precise mathematical definition is the "computational complexity of a problem" which is a measure on how the number of computations grows with a parameter of the problem.