BASICS FOR ODES — Math118, O. Knill

ABSTRACT. This is an overview over the stability of equilibrium points of linear differential equations in the plane.

LINEAR SYSTEMS. A linear differential equation in two dimensions has the form

$$\frac{d}{dt}x(t) = ax + by$$
$$\frac{d}{dt}y(t) = cx + cy$$

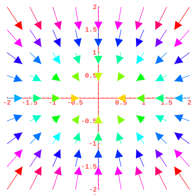
It can be written as $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a vector $\vec{x}$ and a matrix A. We denote the eigenvalues of A with λ_1 and λ_2.

If the eigenvalues are different, one can diagonalize A. In the eigenbasis of A, the matrix is $B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and the differential equation becomes

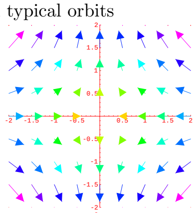
$$\frac{d}{dt}x(t) = \lambda_1 x$$
$$\frac{d}{dt}y(t) = \lambda_2 y$$

with explicit solution $x(t) = e^{\lambda_1 t}x(0), y(t) = e^{\lambda_2 t}y(0)$.

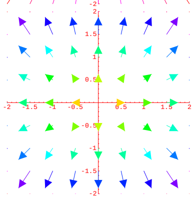
PHASE-PORTRAITS. We plot some vector fields and typical orbits



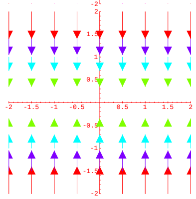
$\lambda_1 < 0$
$\lambda_2 < 0$,
i.e $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$



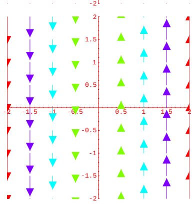
$\lambda_1 < 0$
$\lambda_2 > 0$,
i.e $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$



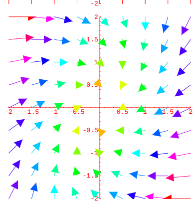
$\lambda_1 > 0$
$\lambda_2 > 0$,
i.e $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



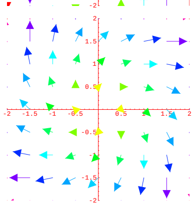
$\lambda_1 = 0$
$\lambda_2 < 0$,
i.e $A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$



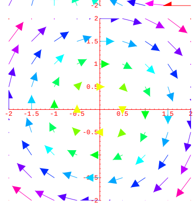
$\lambda_1 = 0$
$\lambda_2 = 0$,
i.e $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$



$\lambda_1 = a + ib, a < 0$
$\lambda_2 = a - ib$,
i.e $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = a + ib, a > 0$
$\lambda_2 = a - ib$,
i.e $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = ib$
$\lambda_2 = -ib$,
i.e $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

AREA PRESERVATION. A differential equation for which we have solutions for all times defines for each t a map T_t in the plane.

We say a differential equation $\frac{d}{dt}x = F(x)$ is **area-preserving** if each of the time t maps T_t is area preserving.

DIVERGENCE. If F is a vector field, we denote by $\text{div}(F)$ the **divergence** of F. It is in two dimensions, where $F(x,y) = (f(x,y), g(x,y))$ given by the formula $\text{div}(F)(x,y) = f_x(x,y) + g_y(x,y)$.

> A differential equation $\frac{d}{dt}x = F(x)$ is area-preserving if and only if $\text{div}(F)(x,y) = 0$ for all points in the plane.

PROOF. By the change of variable formula $\int\int_{T_t(A)} \; dA = \int\int_A |\det(DT(x)| \; dA$, where $DT(x)$ is the Jacobean matrix of the transformation T at x. Because $T_t(x) = x + tF + O(t^2)$, one has $DT_t = I_2 + tDF + O(t^2)$, where I_2 is the identity matrix. We have $DT_t = \begin{bmatrix} 1+ta & tb \\ tc & 1+td \end{bmatrix} + O(t^2)$ we have $\det(DT_t) = 1 + (a+d)t + O(t^2) = 1 + \text{div}(F)t + O(t^2)$. Therefore $\frac{d}{dt}\int\int_{T_t(A)} \; dA = \int\int_A \frac{d}{dt}|\det(DT(x)| \; dA = \int\int_A \frac{d}{dt}(1 + t\text{div}(F)) \; dA = \int\int_A \text{div}(F)(x) \; dA$. (We could get rid of the absolute value because $1 + t\text{div}(F)$ is positive for small t).

2. PROOF. Define $G(x,y,t) = (f(x,y), g(x,y), 1)$ and a tube like region $\{(x(t), y(t), t) \; (x(0), y(0)) \in A, 0 \leq t \leq \tau\}$ in **space-time**. Applying the **divergence theorem** using $\text{div}(G)(x,y,t) = \text{div}(F)(x(t), y(t))$, using the fact that the flux through the cylindrical walls is zero and the flux throught the bottom is $-\text{area}(A)$ and the flux through the top is $\text{area}(T_\tau(A))$ gives $\text{area}(T_\tau(A)) - \text{area}(A) = \int_0^\tau \int_{T_t(A)} \text{div}(F(x(t), y(t))) \; dAdt$. This elegant proof does not need the coordinate change formula.

DISSIPATIVE SYSTEMS. If $\text{div}(F) < 0$ in a region, then area is shrinking. You will explore some of the consequences of dissipation in the homework. Here just an example:

> PROPOSITION. In a region with $\text{div}(F) < 0$, there are no sources or elliptic equilibrium points.

PROOF. If (x_0, y_0) is the equilibrium point, then $\text{div}(F) = \lambda_1 + \lambda_2$. At sources, the real part of both λ_1 and λ_2 are positive. At elliptic equilibrium points, λ_1 and λ_2 are purely imaginary and the sum is 0.

EQUILIBRIUM POINTS. Points, where $F(x,y) = (0,0)$ are called **equilibrium points**. An equilibrium point is called **hyperbolic**, if no eigenvalue has a real part equal to 0. Bifurcations can happen, when an eigenvalue passes through the axes $\text{Re}(\lambda) = 0$. In the hyperbolic case, one can conjugate the system near the equilibrium point to a linear system. This is a continuous version of the Sternberg-Grobman-Hartman theorem.

NULLCLINES. In two dimensions, we can draw the vector field by hand: attaching a vector $(f(x,y), g(x,y))$ at each point (x,y). To find the equilibrium points, it helps to draw the **nullclines** $\{f(x,y) = 0\}, \{g(x,y) = 0\}$. The equilibrium points are located on intersections of nullclines. The eigenvalues of the Jacobeans at equilibrium points allow to draw the vector field near equilibrium points. This information is sometimes enough to draw the vector field **by hand**.

EXAMPLE: COMPETING SPECIES. The system $\dot{x} = x(6 - 2x - y), \dot{y} = y(4 - x - y)$ has the nullclines $x = 0, y = 0, 2x + y = 6, x + y = 5$. There are 4 equilibrium points $(0,0), (3,0), (0,4), (2,2)$. The Jacobian matrix of the system at the point (x_0, y_0) is $\begin{bmatrix} 6 - 4x_0 - y_0 & -x_0 \\ -y_0 & 4 - x_0 - 2y_0 \end{bmatrix}$. Without interaction, the two systems would be logistic systems $\dot{x} = x(6 - 2x), \dot{y} = y(4 - y)$. The additional $-xy$ part is due to the competition. If both x and y become large, then this produce resource problems for both species.

Equilibrium	Jacobean	Eigenvalues	Nature of equilibrium
(0,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6, \lambda_2 = 4$	Unstable source
(3,0)	$\begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = -6, \lambda_2 = 1$	Hyperbolic saddle
(0,4)	$\begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix}$	$\lambda_1 = 2, \lambda_2 = -4$	Hyperbolic saddle
(2,2)	$\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink