N-BODY PROBLEMS — Math118, O. Knill

ABSTRACT. The Newtonian n-body problem influenced the development of mathematics at several occasions. For example, it was the catalisator for the development of calculus or topology. In this section, we look at general facts about the n-body problem like existence of solutions and the nature of singularities like Zeipel theorem which distinguishes collision and noncollision singularities by the convergence of the moment of inertia.

NEWTON EQUATIONS.

Celestial mechanics is the study of the **Newtonian n-body problem**, the study of the differential equations

$$\boxed{m_j \ddot{x}_j = -G \sum_{i \neq j} \frac{m_j m_i (x_j - x_i)}{|x_i - x_j|^3}}$$

The vectors x_j are the positions of the bodies with mass m_j and G is the gravitational constant. If the initial positions and velocities of the bodies are known, then the equations determine the position of the bodies at later times as long as solutions exist. The **phase space** of the system is the 6n-dimensional space $M \times \mathbf{R}^{3n}$, where $M = \mathbf{R}^{3n} \setminus \Delta$ with the **collision set**

$$\Delta = \bigcup_{i \neq j} \Delta_{ij} = \bigcup_{i \neq j} \{ x \in \mathbf{R}^{3n} \mid x_i = x_j \} .$$

HAMILTON EQUATIONS FOR THE N-BODY PROBLEM.

A point (x, y) with $x = (x_1, \dots, x_n), y = (y_1, ..., y_n)$ in the phase space encodes both the positions x_i and the momenta $y_i = m_i \dot{x}_i$ of the bodies. The function

$$H(x, y) = \sum_{j=1}^n \frac{y_j^2}{2m_j} - U(x), \quad U(x) = G \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|}$$

on the phase space is the **energy** of the particle system. One calls it the **Hamiltonian**. The Newton equations can be rewritten as **Hamilton equations**

$$\boxed{\dot{x}_j = \nabla_{y_j} H(x, y), \quad \dot{y}_j = -\nabla_{x_j} H(x, y) \ .}$$

10 CLASSICAL INTEGRALS. An **integral of motion** of a Hamiltonian system is a quantity which is conserved along the orbits.

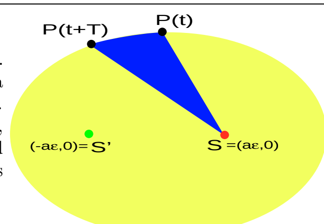
The n-body problem in three dimensions has the **10 classical integrals** of motion:

a) The **total momentum** $Y = \sum_{i=1}^n y_i$.
b) If $Y = 0$, the **position** $C = \sum_{i=1}^n m_i x_i$ of the center of mass.
c) The **total angular momentum** $L = \sum_i x_i \times y_i$.
d) The **total energy** H.

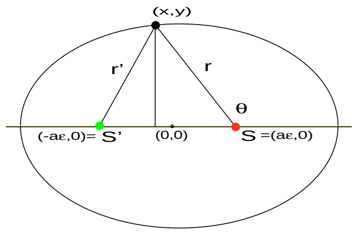
Proofs.
a) Every term in the sum $\dot{Y}$ appears twice but with opposite sign.
b) From $\dot{Y} = C$ follows that if $Y = 0$ then C is constant.
c) $\dot{L} = \sum_{i=1}^n \dot{x}_i \times y_i + \sum_i x_i \times \dot{y}_i$. The second sum is zero because $x_i \times (x_i - x_j) = -x_i \times x_j$ and because each term in the remaining sum appears twice with opposite sign.
d) $\dot{H} = \sum_{i=1}^n H_{x_i} \dot{x}_i + H_{y_i} \dot{y}_i = \sum_{i=1}^n H_{x_i} H_{y_i} - H_{y_i} H_{x_i} = 0$.

THE 2 BODY PROBLEM. After a change of coordinates, one can assume that the center of mass $C = m_1 x_1 + m_2 x_2$ is at the origin. If $q = x_1 - x_2$, then $\ddot{q} = \ddot{x}_1 - \ddot{x}_2 = m_2 G(x_2 - x_1)/|x_2 - x_1|^3 - m_1 G(x_1 - x_2)/|x_2 - x_1|^3 = -(m_1 + m_2) G q / |q|^3$. This is a 1-body problem for a particle with position q and mass $m = m_1 + m_2$ moving in a central field. The angular momentum $L = m x \times \dot{x}$ and the energy $2E/m = \dot{x}^2 + G/|x|$ are conserved quantities.

THE 2. KEPLER LAW. Because $\ddot{x}$ is parallel to x, we get $\dot{L} = 0$. From the conservation of L follows that the vector x stays in a plane, where we can us polar coordinates $x = (r \cos(\theta), r \sin(\theta))$. The constant quantity $L = m r^2 \dot{\theta}$ can be interpreted as df/dt, where f is the area swept over by the vector x. We have derived the "area law", Keplers second law: "the radius vector x passes the same area in the same time."



THE 1. KEPLER LAW. An ellipse with focal points $S' = (-a\epsilon, 0), S = (a\epsilon, 0)$ is the set of points (x, y) whose distances r' and r to S' and S satisfy $r' + r = 2a$. The number ϵ is called the **eccentricity**. From $(2a - r)^2 = r'^2 = r^2 \sin^2(\theta) + (2a\epsilon + r\cos(\theta))^2$, we obtain $r = a(1 - \epsilon^2)/(1 + \epsilon \cos(\theta))$, the **polar form** of the ellipse. Differentiation of this with respect to time, using $\dot{\theta} = L/(mr^2)$ leads to $\dot{r} = a(1 - \epsilon^2) \sin(\theta)(1 + \epsilon \cos(\theta)^{-2} L/(mr^2) = L\epsilon \sin(\theta)/(ma(1 - \epsilon^2))$ and $\ddot{r} = L^2 \epsilon \cos(\theta)/(m^2 a(1 - \epsilon^2) r^2)$. With $n = x/r$, one has $\ddot{x} = (\ddot{x} \cdot n) n$ and $x = nr$ gives $\dot{x} = \dot{n} r + n \dot{r}, \ddot{x} = \ddot{n} r + 2\dot{n}\dot{r} + n\ddot{r}$ so that $\ddot{x} \cdot n = \ddot{n} \cdot n r + 2\dot{n} \cdot n \dot{r} + \ddot{r}$. Using $n \cdot n = 1 \Rightarrow n \cdot \dot{n} = 0, \ddot{n} \cdot n + \dot{n} \cdot \dot{n} = 0$ and $\dot{n} \cdot \dot{n} = \dot{\theta}^2 n^\perp \cdot n^\perp = \dot{\theta}^2 = L^2/(mr^2)^2$, we have



$$\boxed{\ddot{x} \cdot n = \ddot{r} - L^2/(m^2 r^4)}$$

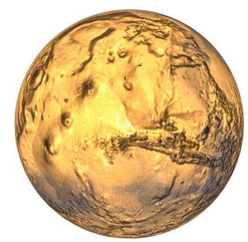
With $1/r^4 = (1/r)(1/r^3) = (1 + \epsilon)/(r^3 a(1 - \epsilon^2))$ and the formula for $\ddot{r}$ we get $\ddot{x} \cdot n = L^2 \epsilon \cos(\theta)/(m^2 a(1 - \epsilon^2) r^2) - (L^2/(m^2))(1 + \epsilon)/(r^3 a(1 - \epsilon^2)) = -L^2/(a(1 - \epsilon^2) r^2$ so that $\ddot{x} = (\ddot{x} \cdot n) n = -L^2/(a(1 - \epsilon^2)) x/r^3 = -Gx/r^3$.

THE 3. KEPLER LAW. If T is the period of the orbit, the the third Kepler law states that T^2/a^3 is constant. Indeed, if $f(t)$ is the area swept by the radial vector from time 0 to time t, then $\dot{f}(t) = L$ implies that the area of the ellipse $\pi a^2 \sqrt{1 - \epsilon^2}$ is equal to LT. From $T = \pi a^2 \sqrt{1 - \epsilon^2}/L$, we get

$$\boxed{T^2/a^3 = \pi^2 a(1 - \epsilon^2)/L^2 = \pi^2/G}$$

The third Kepler law allows to determine the gravitational constant G from the period and the geometry of the ellipse.

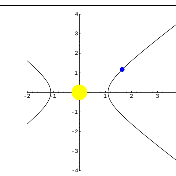
EXAMPLE. A Mars year is 1.88 earth years. How much longer is the length of the major semiaxes of the Mars orbit than the semiaxes of the earth orbit?
Answer: we know $T_{mars}^2/r_{mars}^3 = T_{earth}^2/r_{earth}^3$ so that $r_{mars} = r_{earth}(T_{mars}/T_{earth})^{2/3} = r_{earth} 1.88^{2/3} = 1.523....$ Mars is about one and a half times further away from the sun then the earth.



REMARKS

- To derive the first Kepler law starting with the ellipse is easier than taking off from the differential equations. The later approach is possible but the steps are harder to motivate.
- All Kepler laws crucially depend on the conservation of L.

CULOMB CASE. The case $\epsilon > 1$ corresponds to a negative G, where particles repel each other. The third Kepler law does then no more apply and the curve "ellipse" will be a "hyperbola" in the first law. The second law is unchanged. In this **Coulomb** case of the n-body problem, the total energy is always positive.



OTHER POTENTIALS.

If the interaction potential can be changed to $\ddot{x} = -Gx/r^\alpha$, where α is an integer. We have seen the case $\alpha = 3$. For other α, the first Kepler law still applies. Formula $\dot{\theta} = L/(mr^2)$ still applies. Also the derivation of the formula for $\ddot{x} \cdot n = \ddot{r} - L^2/(m^2r^3)$ is still valid. The left hand side is $-G/r^{\alpha-1}$ which leads to the ordinary differential equation

$$\boxed{\ddot{r} = -G/r^{\alpha-1} + L^2/(m^2r^3)} \qquad (*)$$

for $r(t)$. Knowing $r(t)$ gives then $\theta(t)$ from $\dot{\theta} = L/(mr^2)$. The global behavior depends on the constants G, L. The case $\alpha = 4$ corresponds to the natural Newton interaction in 4 dimensions.
You show in the homework:

In four dimensional space, planetary motion is unstable.

$\alpha = 3$ is the Kepler case with elliptic stable motion.
The case $\alpha = 2$ can physically be realized two massive parallel lines. (The general evolution of two rigid attracting lines in three dimensions is more complicated and form a special case of an interaction of two tops.)

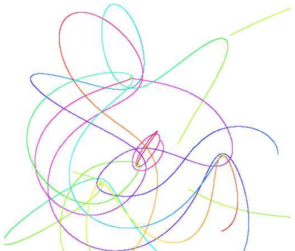
The case $\alpha = 1$ can be realized by the motion of two massive parallel planes. Such planes attract each other with constant force independent of the distance. The equation of motion $\ddot{x} = -G\text{sign}(x)x/|x|$. The three body problem in this case is already interesting. In the case $\alpha = 0$, each coordinate moves according to the harmonic oscillator.

A theorem of Bertant states that only for $\alpha = 3$ (the Kepler case) and $\alpha = 0$ (the harmonic oscillator), all bounded orbits are periodic.

MORE REMARKS.

- To derive the first Kepler law starting with the ellipse is easier than taking off from the differential equations. The later approach is possible but the steps are harder to motivate.
- All Kepler laws crucially depend on the conservation of L.
- In $d \geq 2$-dimensions, one would take the potential $U(x) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^{d-2}}$. In $d = 2$, the natural potential is $U(x) = G \sum_{i<j} m_i m_j \log |x_i - x_j|$.
- A natural regularisation of the singular potential is obtained by replacing the force by $G \cdot (|x|^2 + \epsilon)^{-d/2}$. In that case, one does not have to exclude the collision set Δ.
- The phase space of the system is called with the fancy name **cotangent bundle** of M. Such terminology is not necessary when we deal with particles moving in the open region M of an Euclidean space. However, if one would describe Newtonian particles on surfaces like the sphere or tori, where the interaction potential had to be modified, then the fancier notation is justified. We could for example look at the natural n body problem on a torus or the sphere.
- One would need $(6n - 1)$ integrals of motion to solve the n-body problem explicitly. The 10 classical integrals are not enough to find explicit solutions if $n > 2$. The first mathematical proof of this fact was given by Poincaré in a special case of the three body problem using new qualitative methods.

THE THREE BODY PROBLEM. With 3 or more bodies, the problem becomes chaotic. On the right hand side, you see an orbit computed with the n-body solver "xstar". We will look at the restricted three body problem later in more detail and see in a special situation, the Sitnikov case, that chaos can occur.

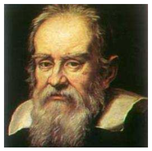


SOME HISTORY.

Aristoteles (384-322 BC) First model of solar system: planets as well the sun move around earth on perfect circles.

Claudius Ptolemaeus (78-150 AC) extended Hipparchus's system of epicycles to explain geocentric theory. Introduced 80 epicycles to explain the motions of sun, moon and 5 planets.

Galileo Galei (1564-1642) discovers Jupiter moons, suns pots etc. Famous for his fight for a Copernican theory with the inquisition. Mathematical work on moments and center of gravity.



Johannes Kepler (1571-1630) builds on the observations of Tycho Brahe. He finds the first and second Kepler law in 1609, the third in 1619.

Joseph-Louis Lagrange (1736-1813) Worked on the 3-body problem, the motion of the moon, and perturbations of comet orbits by the planets as well as the stability of the solar system.

Pierre-Simon Laplace (1749-1827) Investigated the inclination of planetary orbits, studied of planets were perturbed by their moons and the stability of the solar system.

Jean Le Rond d'Alembert (1717-1783) Improved Newton's definition of force in his Trait de dynamique published in 1743. This also contains d'Alembert's principle of mechanics.

George Birkhoff (1884-1944) Tools from probability theory statistical mechanics lead to ergodic theory. An example is Birkhoffs ergodic theorem. Poincare-Birkhoff fixed point theorem.

Jürgen Moser (1928-1999) The "M" in KAM theory. Book with Siegel in Celestial mechanics. Mosers contribution to KAM is the twist map theorem. Worked also on integrable n-body problems.

Hipparchus (190-120 BC) had a moon theory built on epicycles. Still an earth centered system.

Nicolas Copernicus (1473-1543) introduced a heliocentric system as well as secondary epicycles. This is a first step towards perturbation theory (which later would be seen as the Fourier approximation of real motion).

Tycho Brahe (1546-1601) revolutionized astronomy with new instruments and observations. For practical reasons, he used both heliocentric and earth centric coordinate systems.

Isaac Newton (1643-1727) Put celestial mechanics on a solid mathematical foundation and developed calculus simultaneously with Leibniz. Derivation of Keplers laws from basic principles.

Leonard Euler (1707-1783) wrote a 775 page work on the motion of the moon. He won several prizes from the Paris Académie des Sciences in the area of celestial mechanics.

Siméon Denis Poisson (1781-1840) who had Laplace and Lagrange as teachers published in 1808 work on the perturbations of the planets. He used series expansions to derive approximations.

With **Henry Poincaré** (1854-1912) at the end of the 19'th century, the n-body problem was studied with new geometric and topological methods.

Andrey Kolmogorov (1903-1987) The beginning of KAM-theory, which is named after Kolmogorov, Arnold and Moser. Kolmogorov also put probability theory on a solid foundation and worked on a theory of turbulence.

Vladimir Arnold (1937-) Progress on stability questions with perturbative methods (KAM). The concept of **Arnold diffusion** demonstrates a mechanism for instability.