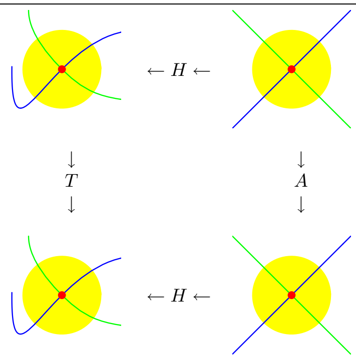
STABLE AND UNSTABLE MANIFOLDS Math118, O. Knill

ABSTRACT. Near a hyperbolic point, one can conjugate the map by its linearization. This conjugation defines local curves through the origin which are invariant. These stable and unstable manifolds intersect in general to form homoclinic points. We will not prove the linearization theorem in class.

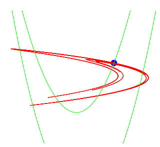
STERNBERG-GROBMAN-HARTMAN LINEARIZATION THEOREM. If $T(x)$ is smooth map with a hyperbolic fixed point x_0, then T is conjugated to its linearization DT near x_0.

Near the fixed point x_0, the dynamics can be computed by first going into a new coordinate system $H^{-1}(x_0)$, applying the linear map A, and undoing the coordinate change by applying H.

More precisely, there exists a small disc D around x_0 and a map H in the plane such that in D the identity $H \circ A(x) = T \circ H(x)$ holds.



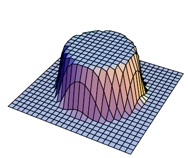
INVARIANT MANIFOLDS. The linear equation $x \mapsto Ax$ has two invariant curves, the lines spanned by the eigenvectors v_i of A. The conjugation defines two invariant curves $r_i(t) = H(tv_i)$ through a hyperbolic fixed point. These curves are called **stable and unstable manifolds** of the hyperbolic fixed point. The picture shows the stable and invariant manifolds for one of the fixed points of the Henon map. The unstable manifold lies in the attractor. Note that the unstable manifold of $T(x,y) = (1 - ax^2 + y, bx)$ is the stable manifold for $T^{-1}(x,y) = (y/b, (x - 1 + ay^2/b^2)$.



Here is the proof of the linearization theorem in its simplest case. The conjugation can actually be proven to be smooth too. The theorem had first be proven by S. Sternberg in 1958 (smooth conjugation for smooth T) and P. Hartman in 1960 (C^1 conjugation for C^2 maps T). The proof (not done in class) is not so easy and requires the language of linear operators.

PROOF PART 0: Some notations and preparations.
The proof needs works in any dimensions, so x is now a vector in n-dimensional space $X = R^n$. Write $C(X,X)$ for the linear space of all continuous maps from X to X. The norm on this space is defined as $||f|| = \sup_{x \in X} |f(x)|$. The norm of a linear operator U from $C(X,X)$ to $C(X,X)$ is defined as $||U|| = \sup_{||f||=1} ||U(f)||$. A linear map is called a contraction if $||U|| < 1$. If U is a contraction, then $(1-U)$ is invertible: the inverse is given by a geometric series $(1-U)^{-1} = \sum_{n=0}^\infty U^n$. For a hyperbolic matrix A, we write $X = E_+ \oplus E_-$, where E^+ is the linear space spanned by the eigenvectors of A belonging to the eigenvalues $|\lambda_i| < 1$ and E^- is the space spanned by the eigenvectors to A belonging to the eigenvalues $|\lambda_i| > 1$.

PROOF PART I: Reduction to a global conjugation problem.
Take first a smooth scalar function $\phi_\epsilon(x)$, which satisfies $\phi_\epsilon(x) = 1$ for $|x - x_0| > 2\epsilon$ and $\phi_\epsilon(x) = 0$ for $|x - x_0| < \epsilon$ (see picture to the right). The map $S = T + \phi_\epsilon \cdot (A - T)$ is equal to T for $|x - x_0| < \epsilon$ and equal to A for $|x - x_0| > 2\epsilon$. If can write $S(x) = Ax + f(x)$, where f is a smooth map satisfying $||f'||_\infty \to 0$ for $\epsilon \to 0$. Using this surgery, we can solve a global problem.



PROOF PART II: The conjugating equation and its linearization.
The aim is to show that S is conjugated by a map $H(x) = x + h(x)$ to the linear map A if $S = A + f$ if $||f'||_\infty$ is small enough. Remember that $f' = Df$ is the Jacobean matrix of f. The condition $H \circ A(x) = S \circ H(x)$ can be rewritten with $S(x) = Ax + f(x), H(x) = x + h(x)$ as

$$h(A(x)) - Ah(x) = f(x + h(x)) .$$

It is an equation for the unknown map $h \in C(X,X)$. We first consider the **linearized problem**

$$(Lh)(x) := h(A(x)) - Ah(x) = f(x) .$$

PROOF PART III: Solving the linearized problem.
We can decompose the problem into two parts

$$h_\pm(A(x)) - Ah_\pm(x) = f_\pm(x) ,$$

where $h = h_+ + h_-, f = f_+ + f_-$ is the decomposition satisfying $f_\pm, h_\pm \in E^\pm$. The linear map on continuous functions on the plane $U : C(X) \mapsto C(X), h \mapsto h(A)$ as well as its inverse U^{-1} have norm $||U|| = ||U^{-1}|| = 1$. We write $Af = A_+f_+ + A_-f_-$. Because

$$\begin{aligned} ||(U - A_+)^{-1}|| &= ||U^{-1} \sum_{n=0}^\infty A_+^n U^{-n}|| \leq \frac{1}{1-\lambda} \\ ||(U - A_-)^{-1}|| &= ||A_-^{-1} \sum_{n=0}^\infty A_-^{-n} U^n|| \leq \frac{\lambda}{1-\lambda} < \frac{1}{1-\lambda} \end{aligned}$$

with $\lambda = \max\{||A_+||, ||A_-^{-1}||\} < 1$, we can find h using the formula

$$h = h_+ + h_- = (U - A_+)^{-1} f_+ + (U - A_-)^{-1} f_- .$$

PROOF PART IV: Solving the nonlinear problem.
Define $\Phi(h)(x) = f(x + h(x)) - f(x)$. We need to solve the equation

$$Lh = \Phi h + f$$

in for the unknown h in $C(X)$. The solution to this equation $(L^{-1}\Phi - 1)h = L^{-1}f$ is

$$h = (1 - L^{-1}\Phi)^{-1} L^{-1} f$$

if $1 - L^{-1}\Phi$ is invertible. Sufficent to invertibility is that $L^{-1}\Phi$ is a contraction. This is indeed the case if ϵ is small that is if $||f'||_\infty$ is small:

$$||(L^{-1}\Phi)h_1 - (L^{-1}\Phi)h_2|| \leq \frac{1}{1-\lambda} \cdot ||\Phi h_1 - \Phi h_2||_\infty \leq \frac{1}{1-\lambda} \cdot ||f'||_\infty \cdot ||h_1 - h_2|| .$$

COMPUTATION OF MANIFOLDS. The stable and unstable manifolds of a hyperbolic fixed point can be computed using power series. This calculation is due to Francescini and Russo. To get one of the manifolds, construct a curve $r(t) = (x(t), y(t))$ satisfying $r(0) = (x_0, y_0)$ and

$$T(x(t), y(t)) = (1 - ax(t)^2 + y(t), bx(t)) = (x(\lambda t), y(\lambda t))$$

for all $t \in R$. Here λ is an eigenvalue of the Jacobean matrix at the fixed point. Because $y(\lambda t) = bx(t)$, it is enough to calculate $x(t)$. With a Taylor series $x(t) = \sum_{n=0}^\infty a_n t^n$, the invariance condition $1 - ax(t)^2 + y(t) = x(\lambda t)$ or equivalently $x(\lambda t) + ax(t)^2 - bx(\lambda^{-1}t) = 1$ becomes

$$\sum_{n=0}^\infty [a_n \lambda^n - ba_n \lambda^{-n} + a \sum_{j=0}^n a_j a_{n-j}] t^n = 1 .$$

This equation allows to calculate the Taylor coefficients a_n recursively. Comparing coefficients of t^n gives $a(a_0 a_n + a_1 a_{n-1} + ... + a_{n-1} a_1 + a_n a_0) - b\lambda^{-n} a_n = -\lambda^n a_n$ and so

$$a_n = \frac{a(a_1 a_{n-1} + ... + a_{n-1} a_1)}{-\lambda^n - 2aa_0 + b\lambda^{-n}}$$

once $a_0, ..., a_{n-1}$ are given. The first coefficient a_0 is just x_0. Because a_1 satisfies $2aa_0a_1 - b\lambda^{-1}a_1 = a_1\lambda$, it can be chosen arbitrary like $a_1 = 1$. For the parameters $a = 1.4, b = 0.3$ the unstable manifold is $r(t) = (0.631354 + t - 0.25986t^2 + ..., 0.189406 - 0.155946t - 0.0210654t^2 + ...)$, the stable manifold is $r(t) = (0.631354 + t + 0.13278t^2 + ..., 0.189406 + 1.92374t + 1.63796t^2 + ...)$.

HOMOCLINIC POINTS. The intersection points of stable and unstable manifolds different from the fixed point itself are called **homoclinic points**. It has been realized already by Poincaré that the existence of homoclinic points produces a horrible mess. We will see why soon.