HIGHER DIMENSIONAL CA

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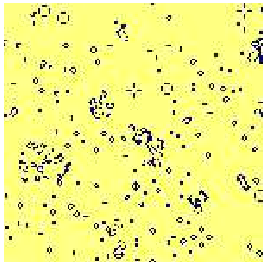
ABSTRACT. We look at some higher dimensional automata like the game of life or lattice gas automata. Note that 2 hours after this lecture, unix time is 1111111111 = Fri, 18 Mar 2005 01:58:31.

HIGHER DIMENSIONAL AUTOMATA. Everything said before can be generalized to higher dimensions. Lets restrict to two dimensions. The space is $X = A^{Z^2}$. It consists of elements $x_{n,m}$, where (n, m) are the coordinates. Define the shifts $\sigma_1(x)_{n,m} = x_{n+1,m}, \sigma_2(x)_{n,m} = x_{n,m+1}$. A continuous map on X which commutes with both σ_i is called a Cellular automaton. We have $T(x)_n = \phi(x_m)$ with $n - m$ in some finite set F. The composition of two CA is a CA. A distance is defined as $d(x, y) = 1/(n + 1)$ if $x_k = y_k$ for $|k| \leq n$ and $x_l \neq y_l$ for some $|l| = n$, where $|(i, j)| = |i| + |j|$.

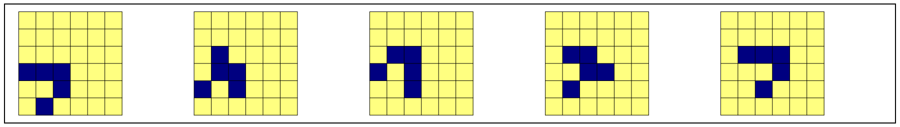
GAME OF LIFE. One of the most famous automaton is **Conways game of life**. A dead cell comes alive if and only if it has three neighbors. A live cell dies if it has less then 2 ore more than 3 neighbors.



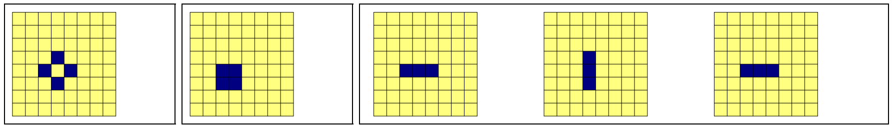
SPECIAL SOLUTIONS. A configuration x has **compact support** if there are only finitely many cells which are alive. Examples of solutions with compact support are gliders, stones and blinkers.

The picture to the right shows life after a random initial condition, after having iterated for 500 iterations.

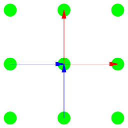
GLIDERS. Solutions which satisfy $T^n(x) = \sigma^v(x)$ for integer n and $v = (v_1, v_2)$ are called **gliders**. Gliders travel with velocity v/n. If x is a glider, then $T^n(x)$ converges to 0.



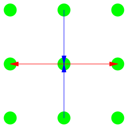
PERIODIC SOLUTIONS. If $T^n(x) = x$, then x is called a periodic solution of T. The left two configurations below show fixed points called "stones". We also see a periodic two orbits called "blinker".

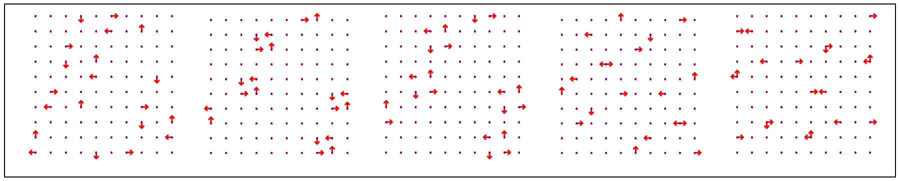


THE HPP MODEL. is a simple deterministic two-dimensional cellular automata designed by Hardy,Pazzis and Pomeau in 1972. Its aim to have a simple toy model to simulate the Navier Stokes equations. The automaton has a color for each of the possible particle configurations. There can be maximally 4 particles at the same spot. One assigns a letter to each of the 16 configurations.

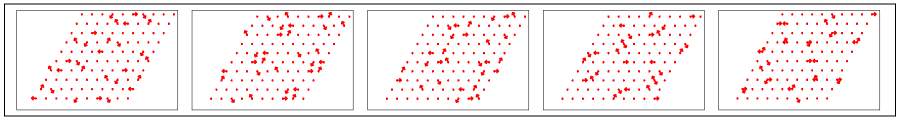


Particles always point away from the origin. Either there is a particle in one of the four directions, or there is not. Once can code each color with a code like $(n, w, s, e) = (1, 1, 0, 1)$ The rules are designed such that particles move freely. For example, if if $x_{n,m} = (0, 0, 0, 1)$ and all other nodes satisfy $x_{i,j} = (0, 0, 0, 0)$, then $x_{n+1,m} = (0, 0, 0, 1)$. A particle has moved from node (n, m) to node $(n + 1, m)$. If particles collide with a right angle, they will scatter as if they would pass through each other. If they hit head on, both directions change by 90 degrees.





HEXAGONAL LATTICE GAS CA. Designed by Frisch,Hasslacher and Pomeau in 1985. The rules are designed to conserve particle number and momentum at each vertex. Additionally, there is a random number generator, when particles collide head on. The possible directions in which the particle pair can scatter is chosen randomly. Also this lattice gas automaton conserves particle numbers as well as momentum of the particles.



ATTRACTOR. The image $X_1 = T(X_0)$ of the set of all configurations $X_0 = A^{Z^d}$ is a T invariant subset. The image $X_2 = T(X_1)$ is invariant too etc. We obtain a nested sequence of subsets $X_0 \supset X_1 \supset X_2 ....$ The limit $X = \bigcap_k X_k$ is called the **attractor** of the cellular automaton. It is a closed T-invariant subset and $T(X) = X$.

WHERE DO CA BELONG?

Space	Time	States	Object
Continuous	Continuous	Continuous	Partial differential equations
Continuous	Discrete	Continuous	Maps on function spaces
Discrete	Continuous	Continuous	Coupled differential equations
Discrete	Discrete	Continuous	Coupled map lattices
Discrete	Discrete	Discrete	Cellular automata

PDE Example: $\frac{d}{dt}u(x, t) = f(u(x, t), \frac{d}{dx}u(x, t))$.
Maps on functions: $u(x, t + 1) = f(u(x, t))$.
Coupled differential equations: $\frac{d}{dt}u(n, t) = f(u(n - 1, t), u(n, t), u(n + 1, t))$
Coupled map lattices: $u(n, t + 1) = f(u(n - 1, t), u(n, t), u(n + 1, t))$ with $u(t, x)$ real.
Cellular automata: $u(n, t + 1) = f(u(n - 1, t), u(n, t), u(n + 1, t))$ with $u(t, x)$ finite.

HISTORY. Numerical treatments of ODE's and PDE's leads to CA: Example: the heat equation $u_t = u_{xx}$ leads to a difference equation $u(t + 1, x) - u(t) = cu(t, x + 1) - 2u(t, x) + u(t, x - 1)$ which becomes a CA, when $u(t, x)$ takes finitely many values only. If the PDE is translational invariant, the discretisation is a CA with an alphabet of $1/\epsilon$ elements, if the computing accuracty is ϵ. Difference methods for PDEs were used since a long time, at least since 1920 (L.F. Richardson), and research on it exploded during WW2 and when the first computers appeared (i.e. the first electronic computer ENIAC in 1945). John von Neumann seemed have introduced CA in these years. Ulam claims to have found CAs first in "Adventures of a Mathematician" p.285: "my own simple minded model". 1936 Turing machines are shown to be able to do all computations. A Turing machine with n states and a tape alphabet of k symbols is a special cellular automaton with an alphabet of $n + k$ letters.

1950 Idealized models of biological systems were studied using CA. Ulam and von Neuman called this "nearest neighbor-connected cellular spaces". Source: From Cardinals to Chaos, Ed: Necia Grant Cooper, Cambridge University Press.

1969 Gustav Hedlund considered in the mid 50ies "shift commuting block maps". see "Endomorphisms and automorphisms of the shift dynamical systems" Math. Systems Theory 3, p.320-375, (1969). Hedlund got his PhD at Harvard in 1930.

1970 Conway article on the "game of life" in the Scientific American 223, (October 1970): 120-123. The name CA had already been coined, like in "Essays on cellular automata Ed. Arthur W. Burks, 1970.

2004 MathSciNet shows 3328 papers authored on Cellular automata.