LIENHARD SYSTEMS — Math118, O. Knill

ABSTRACT. For a certain class of differential equations called Lienard systems, one can prove the existence of a stable limit cycle. An example is the van der Pol oscillator.

LIENHARD SYSTEMS. A differential equation

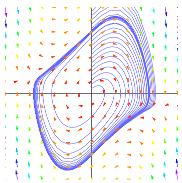
$$\frac{d^2}{dt^2}x + F'(x)\frac{d}{dt}x + G'(x) = 0$$

is called a **Lineard system**. With $y = \frac{d}{dt}x + F(x)$, $G'(x) = g(x)$, this is equivalent to

$$\begin{aligned} \frac{d}{dt}x &= y - F(x) \\ \frac{d}{dt}y &= -g(x) \ . \end{aligned}$$

VAN DER POL EQUATION. If $F(x) = c(x^3/3 - x)$ and $G(x) = x^2/2$, we have **van der Pol equation**

$$\frac{d^2}{dt^2}x + c(x^2-1)\frac{d}{dt}x + x = 0$$



Physically, one has a harmonic oscillator $\frac{d^2}{dt^2}x + x = 0$ for $c = 0$. For $c > 0$, some velocity and space dependent force $c(x^2-1)\frac{d}{dt}x$ is added. This force is accelerating the oscillator, if $x^2 < 1$, it is slowing down the oscillator if $x^2 > 1$. For large c, one calls the oscillator a **relaxation oscillator** because the stress accumulated during a slow buildup is relaxed during a sudden discharge.

THEOREM (Lienard) Assume F and g are smooth odd functions such that $g(x) > 0$ for $x > 0$ and such that F has exactly three zeros $0, a, -a$ with $F'(0) < 0$ and $F'(x) \geq 0$ for $x > a$ and $F(x) \to \infty$ for $x \to \infty$. Then the corresponding Lienard system has exacactly one limit cycle and this cycle is stable.

REMARK ON THE FIXED POINT $(0,0)$: Because g is odd with $g(x) > 0$ for $g \geq 0$, we have $g'(0) \geq 0$. The Jacobean matrix

$$\begin{bmatrix} F'(x) & 1 \\ -g'(x) & 0 \end{bmatrix}$$

has the eigenvalues $\lambda_{1,2} = (-F'(x) \pm \sqrt{F'(x)^2 - 4g'(x)})/2$. At the fixed point, the real part of these eigenvalues is positive because by assumption $F'(0) < 0$ and $|\sqrt{F'(x)^2 - 4g'(x)})| \leq |F'(x)|$ since $g'(0) \geq 0$. We see that the fixed point 0 is repelling.

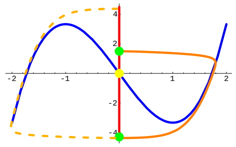
SOME REMARKS. Stable limit cycles appear in ecological, biological as well as mechanical systems. They are relevant because they are in general stable under small changes of the system.

From 1920 to 1950, research on nonlinear oscillations florished. The work was initially motivated by the development of radio and vacuum tube technology, where one realized that many oscillating circuits could be modeled by Lienard systems. This has been applied to many other situations. For example, one has also modeled the periodic firing of nerve cells driven by a constant current using van der Pol type differential equations.

Balthasar Van der Pol (1889-1959) was a Dutch electrical engeneer. He started his investigation on the van der Pol equation in 1926 and also studied versions with periodic forcing term, where chaotic motion can occur.

Lienards theorem was found and published in Russian by **Lienard** in 1958. For the proof of the Lienards theorem, we followed the proof given in the book "Differential equations and Dynamical systems" of Lawrence Perko.
A nice discussion can also be found in the book "Nonlinear dynamics and Chaos" by Steven Strogatz.
For historical facts mentioned in this section, we used "Writing the History of Dynamical Systems: Longe Duree and Revolution, Disciplines and Cultures" by David Aubin and Amy Dahan Dalmedico in Historia Mathematica 29, 2002. One should note also **Mary Cartwright** (1900-1998), who was making important contributions to the theory of nonlinear oscillations and discovered many phenomena later known as chaos (when the oscillator is driven, it becomes chaotic).

PROOF OF LIENHARDS THEOREM.
Draw in the $xy-$ plane the graph of the function $x \to F(x)$. On this graph, the vector field is vertical. It is called a **nullcline**. For $x > 0$ we have $\frac{dy}{dx} < 0$. On the y-axes, the vector field is horizontal because $g(0) = 0$. The y-axes is also a nullcline.



Consider an orbit which starts at $(0, y_0)$ on the positive y axes. It goes to the right because $g(x) > 0$ for $x \geq 0$. Because $g(x) > 0$ for $x > 0$, the orbit also moves down. It has to hit the graph of F. It intersects that nullcline at a point $(x_1, 0)$ with positive vertical velocity and enters the region, where $\frac{dy}{dx} < 0$. It must then go to the left and hit again somewhere the y axes horizontally in some point $(0, y_1) = (0, -S(y_0))$.
Because the differential equations are invariant under the transformation $(x,y) \mapsto (-x,-y)$, we can analyze the fate of the orbit on the left half plane in the same way as on the right plane.

A limit cycle exists if the map $y_0 \to S(y_0)$ has a fixed point. Alternatively, we can express this that the "energy" $H(x,y) = y^2/2 + G(x)$ is the same at $(0, y_0)$ and $(0, y_1)$. The idea of the proof is to determine the energy gain along the orbit and to see that only for one single orbit, the energy is conserved.

Compute

$$\frac{d}{dt}H(x,y) = y\frac{d}{dt}y + g(x)\frac{d}{dt}x = -F(x)g(x)$$

If $F(x(t))$ were positive on the entire trajectory from $(0, y_0)$ to $(0, y_1)$, then $H(0, y_1) - H(0, y_0)$ is positive. It must therefore cross the graph of F at a point, where $F(x) > 0$. The theorem is proven if we can show the following statement about the energy difference

$$\Delta(y_0) = H(0, S(y_0)) - H(0, y_0)$$

depending on the intersection point $(x_1, F(x_1))$ with the null cline.

If $x_1 \leq a$, then $\Delta(y_0) > 0$. For y_0 such that $x_1 > a$, $\Delta(y_0)$ is a monotonically decreasing function for y_0. and $\Delta(y_0) \to -\infty$ for $y_0 \to \infty$.

As a consequence, there exists then exactly one point y_0, where the energy gain is zero. This point y_0 belongs to a limit cycle. The rest of the proof is devoted to the verification of the above claim.

(i) $\Delta(y) > 0$ **if** y_0 **is such that** $x_1 \leq a$.
Note that $F(x)$ is negative in the interval $[0, a]$. If $x_1 \leq a$, then $x(t) \leq a$ until we hit the y axes again. But since then $F(x(t)) < 0$ and $g(x) > 0$ for $x > 0$, we have $\frac{d}{dt}H(x,y) = -F(x)g(x) > 0$. The energy gain is positive.

(ii) **The monotonicity claim for** $x_1 \geq a$.
Let $A(y_0)$ be the path $(x(0), y(0)) = (0, y_0)$ and $(x(T), y(T)) = (0, y_1)$. From $\frac{d}{dt}H(x,y) = -F(x)g(x)$ we obtain

$$\Delta(H)(y_0) = \int_A -F(x(t))g(x(t)) \ dt = -\int_A F(x(y)) \ dy = \int_A -\frac{F(x)g(x)}{y - F(x)} \ dx \ .$$

Split the path A into a path A_1 from $(0, y_0)$ to $x(t) = a$, a path A_2 which is the continuation until $x(t) = a$ again and into a path A_3 until $(0, y_1)$. Along A_1 and A_3, we can parametrize the curve by x instead of t, along A_2, we can use the parameter y.

We see that increasing y_0 increases $y(t)$ and so decreases the integral $\Delta_1(H)(y_0) = \int_0^a -\frac{F(x)g(x)}{y-F(x)} \ dx$ along A_1. On A_3 increasing y_0 decreases $y(t)$ which decreases the integral $\Delta_3(H)(y_0) = \int_0^a \frac{F(x)g(x)}{y-F(x)} \ dx$ along along A_3. Along A_2, use y as the variable. Increasing y_0 pushes the path A_2 to the right so that $F(x(t))$ is increasing and the integral $\Delta_2(H)(y_0) = -\int_{y_2}^{y_3} F(x(y)) \ dy$ is decreasing. The sum $\Delta(H)(y_0) = \Delta_1(y_0) + \Delta_2(y_0) + \Delta_3(y_0)$ is decreasing in y_0.

(iii) **The limit** $y_0 \to \infty$.
To see that $\Delta(y_0)$ goes to $-\infty$ for $y_0 \to \infty$, we split an orbit into paths B_1, B_2, B_3 in the same way as A_1, A_2, A_3 but where the value of a has been replaced by $a + 1$. The integrals along B_1 and B_3 are bounded by a constant independent of y_0, while the integral along B_2 is bigger or equal to $F(a+1)$ times the y differences of the two points, where $x(t) = a + 1$. This difference goes to $-\infty$ for $y_0 \to \infty$. So, the energy gain along the sum of the paths B_1, B_2, B_3 goes to $-\infty$ for $y_0 \to \infty$.