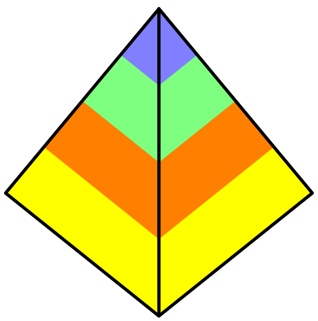
2/2/2004 INTRODUCTION — Math118, O. Knill

ABSTRACT. We discuss the methodology and organization of the course.

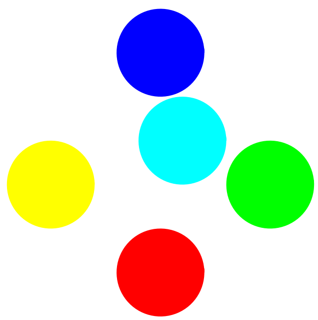
The subject. Dynamical system theory has matured into an independent mathematical subject. It is linked to many other areas of mathematics and has its own AMS classification which is 37-xx. The subject has grown so fast, that already specific subareas of dynamical systems like one-dimensional dynamics or ergodic theory have become independent research areas. As in other mathematical subjects, like topology, geometry or analysis which have "settled down", the teaching of the subject from the bottom up needs a lot of time. It makes more sense to study the subject by picking a few interesting subtopics.

The case method. This course is taught with an adaptation of the 'case method'. Each week, we pick a topic and use it to discuss some aspect of dynamical systems theory. The advantage of the 'case method' approach is that one can start early with mentioning open research topics. Furthermore, there are frequent fresh starts. We used this style for a course called "Mathematical Chaos Theory" in 1994 at Caltech, where an integral part were computer demonstrations using Mathematica and special software. It was also the first course, where I had used a course web-site.

The "case method" style has been used in Mathematics for a long time. Examples are the booklet **pearls of number theory** by Khinchin or Bowen's lectures in dynamical systems theory. The case method is a traditional Russian presentation style which can be found in many books. It is also used in research summer schools, where the breakup into different subjects and lectures comes naturally.



Systematic approach



Case approach

The history of a mathematical subject. Each part of the course has its own theme and flavor and is labeled by the name of a "protector", which is either a mathematician or physicist. We try to keep each subject independent of the others but of course, we will cross reference and relate to older topics. We also aim to give a glimpse into the history and gossip of the given subject. Because many different topics are covered, you will be able to get an idea, what dynamical systems is about and pick your favorite theme for a final project. which can either be of experimental or theoretical nature.

The level of difficulty. The course should be attractive for people who are interested in the applications of dynamical systems theory as well as for students, who want to see more mathematics beyond calculus. Some of the mathematical facts mentioned in class will be proven in full mathematical rigor and illustrated with live experiments in class. Participants of the course will be provided tools to experiment using online applications, computer algebra systems or using their own favorite programming language. No programming knowledge is required. More theoretically inclined or application oriented students will be given the opportunity to read some hand-picked survey articles if they wish.

Other fields Many introductory books on dynamical systems theory give the impression that the subject is about iterating maps on the interval, watching pictures of the Mandelbrot set or looking at phase portraits of some nonlinear differential equations in the plane. This is far from the reality. The topic can be seen as an interdisciplinary approach to many mathematical and nonmathematical areas. The field has matured and is successfully used in other fields like game theory, it is used to approach difficult unsolved problems in topology, and helps to see number theoretical problems with different eyes. There is hardly any mathematical field, which is not involved. For example: iterating smooth map or evolving smooth flows on manifolds is rooted in geometry, a sequence of independent random variables in probability theory can be modeled as a Bernoulli shift, the law of large numbers a special case of the ergodic theorem, the learning process in artificial intelligence can be seen as a discretized gradient flow. Dynamical systems are used heavily in number theory. For example, to understand the frequency of decimal digits occurring in the real number $\pi = 3.14159...$, where a dynamical systems approach looks the most promising one. The practical applications of the theory of dynamical systems are enormous: it ranges from medical applications like bifurcations of heartbeat patterns to explain the synchronous rhythmic flashing of fireflies. And then there are the obvious applications in population dynamics, fluid dynamics, quantum dynamics or statistical mechanics.

Prerequisites. To follow this course, a one semester multi-variable calculus like math21a, applied math21a, math23b, as well as a one semester of linear algebra course like math21b, applied math23b, math23b is required.

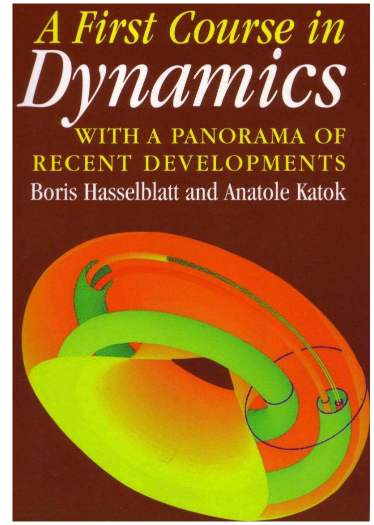
Exams. We plan to do several small quizzes. This, the homework, a final project and participation will make up the grade.

Syllabus. The difficulty and pace of the course will somehow be adjusted according to the audience. For a modular course like that, the theme structure allows an easy adaption of the pace.

- **1. Week: Introduction**.
  - What are dynamical systems
  - Organization of the course
  - Examples of dynamical systems
- **2. Week: Feigenbaum: maps in one dimensions**.
  - Maps on the interval
  - Periodic points and their stability.
  - Bifurcation of periodic points
  - The dynamical zeta function
  - Invariant measures
  - The Lyapunov exponent
- **3. Week: Henon: maps in two dimensions**.
  - Area preservation
  - Periodic points and their nature
  - Stable manifold theorem and homoclinic points
  - Construction of stable manifolds
  - Lyapunov exponents and random matrices
  - Definitions of chaos
- **4. Week: Hilbert: Differential equations in two dimensions**
  - Differential equations in the plane and torus
  - Poincare-Bendixon theorem
  - Limit cycles
  - Hopf bifurcations
  - The Hilbert problem on limit cycles
- **5. Week: Lorentz: ODEs in higher dimensions**
  - Differential equations in space
  - The attractor in the Lorentz system
  - Forced oscillators
  - Lyapunov functions for ODE's
  - Strange attractors

- **6. Week: Birkhoff: billiards**
  - Billiards
  - The variational setup
  - Existence of periodic points
  - Polygonal billiards
  - Chaos for the stadium billiard
- **7. Week: Hedlund: cellular automata**
  - Curtis-Hedlund-Lyndon theorem
  - Topological entropy for CA
  - Attractors
  - Higher dimensional automata
  - Special solutions
- **8. Week: Mandelbrot: maps in the complex plane**
  - Mandelbrot and Julia sets
  - Some topological notions
  - Connectivity of Mandelbrot set
  - Iterations of quaternions
  - Complex Henon map
- **9. Week: Bernoulli: subshifts of finite type**
  - Bernoulli shift
  - Subshifts of finite type
  - Normal numbers and randomness
  - Entropy
  - Normal numbers and randomness
- **10. Week: Weyl: dynamical systems in number theory**
  - Irrational rotation on the torus
  - Unique and strict ergodicity
  - Continued fractions
  - Diophantine problems
- **11. Week Poincare: many body problems**
  - The equations of the n-body problem
  - Integrals and the solution of the 2 body problem
  - The Sitnikov problem
  - The role of singularities
  - Informal description of non-collision singularities
- **12. Week: Einstein: geodesic flows**
  - Geodesic flows on the torus
  - Integrability and examples
  - Wave fronts and Huygens principle
  - Caustics
- **13. Week: Review**
  - Review
  - Open problems in dynamical systems
  - Projects

The book. It is important to have a 'second opinion' on things. We will not follow a book but the "First course in Dynamics, with a panorama of recent developments" by Boris Hasselblatt and Anatole Katok comes closest. It is written by leading experts in the area of dynamical systems.



More literature suggestions can be found on the course web-site.

The website. All of the material will be available on the course website:

http://www.courses.fas.harard.edu/ math21b.