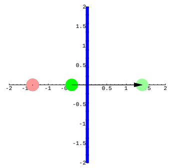
BIFURCATIONS **Math118, O. Knill**

ABSTRACT. Equilibrium points can bifurcate. One distinguishes **pitchfork bifurcation** and **blue-sky bifurcation**, which were already knew in the one-dimensional setting. In two dimensions, where limit cycles can occur, it can happen that an equilibrium point produces a limit cycle. This is called the **Hopf bifurcation**.

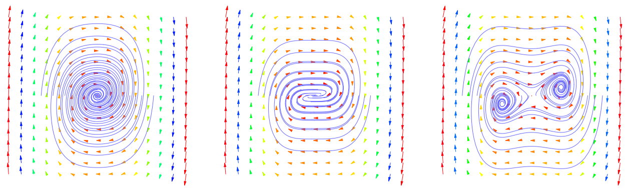
BIFURCATIONS OVERVIEW. If an eigenvalue of the Jacobean DF at an equilibrium point (x_0, y_0) crosses the y-axes, the stability of the equilibrium point changes. As in the discrete case, this is called a **bifurcation**. What possibilties are there? Besides the **pitch-fork** and **blue-sky** bifurcations, we already already in one dimensions, there are now possibilities which are not known in one dimensions. One is called **Hopf bifurcation**, which is the birth of **limit cycles**.

PITCHFORK BIFURCATION. If both eigenvalues were initially in the left half plane and one eigenvalue moves over to the right half plane, then a stable sink becomes a hyperbolic point. This bifurcationis usually associated to the creation of two new stable equilibrium points. This is called a **pitchfork bifurcation**. We know the one-dimensional version of this bifurcation already.

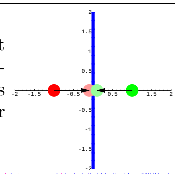


Example: $c = 0$ is a bifurcation parameter for

$$\begin{aligned} \frac{d}{dt}x &= y - 0.3 * x \\ \frac{d}{dt}y &= cx - x^3 \end{aligned}$$

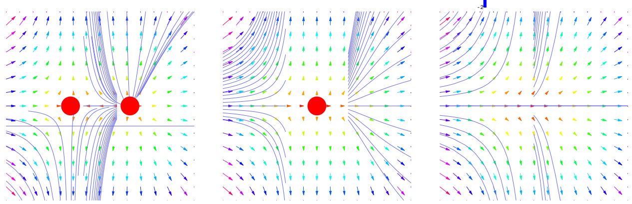


BLUE SKY BIFURCATION. It can happen that a hyperbolic equilibrium point collides with a stable or unstable equilibrium point and disappears. The opposite is also possible. Out of the blue, a parabolic equilibrium appears and splits into two equilibrium points. This is called the **saddle node bifurcation** or **blue-sky bifurcation**.

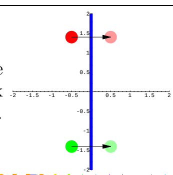


Example: $c = 0$ is a bifurcation parameter for

$$\begin{aligned} \frac{d}{dt}x &= c + x^2 \\ \frac{d}{dt}y &= -y \end{aligned}$$

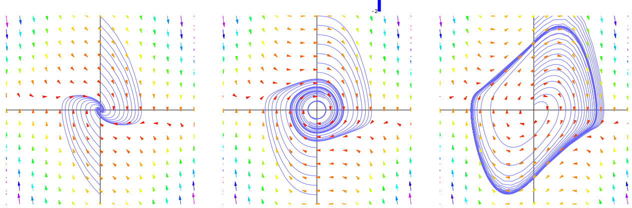


If both eigenvalues are on the left hand side and cross at the same time, we have a **Hopf bifurcation**. In this case, a stable sink becomes an unstable source and a limit cycle will appear nearby.



Example:

$$\begin{aligned} \frac{d}{dt}x &= y \\ \frac{d}{dt}y &= -x - (x^2 - c)y \end{aligned}$$

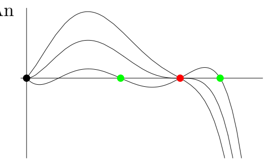


MORE BIFURCATIONS WITH LIMIT CYCLES (what follows will not be quizzed). These bifurcations above started with critical points and led to limit cycles. With limit cycles, there are more possibilites:

PITCH-FORK BIFURCATION FOR LIMIT CYCLES.
A stable limit cycles can change stability, become unstable and produce two limit cycles. This is called the **saddle node bifurcation** for limit cycles. An example is given in polar coordinates by

$$\begin{aligned} \frac{d}{dt}r &= r(r(1-r)^3 + c((r-1)^3 + (r-1))) \\ \frac{d}{dt}\theta &= \alpha + r^2 \end{aligned}$$

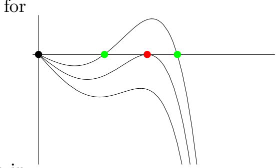
(It can using the formula $x = r\cos(\theta), y = r\sin(\theta)$ be rewritten as a system in the x, y coordinates.)



SADDLE NODE BIFURCATION FOR LIMIT CYCLES.
The sudden appearance of limit cycles is called **saddle node bifurcation** for limit cycles. An example is given in polar coordinates by

$$\begin{aligned} \frac{d}{dt}r &= cr + r^3 - r^5 \\ \frac{d}{dt}\theta &= \alpha + r^2 \end{aligned}$$

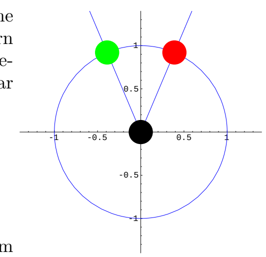
(It can using the formula $x = r\cos(\theta), y = r\sin(\theta)$ be rewritten as a system in the x, y coordinates.)



INFINITE PERIOD BIFURCATION.
A blue-sky bifurcation for equilibrium points can appear on a limit cycle. The limit cycle will become the stable and invariant manifolds of the newly born hyperbolic points. This bifurcation is called **infinite period** bifurcation because the limit cycle period will satisfy $T \to \infty$. An example is given in polar coordinates by

$$\begin{aligned} \frac{d}{dt}r &= r(1 - r^2) \\ \frac{d}{dt}\theta &= c - \sin(\theta) \end{aligned}$$

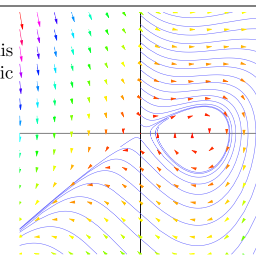
The system has an invariant circle for all c but for $c = 1$, there is an equilibrium point on the circle.



HOMOCLINIC BIFURCATION.
An equilibrium point can collide with a limit cycle and "open" it up. This bifurcation is called a **homoclinic bifurcation**. An example of a homoclinic bifurcation happens for

$$\begin{aligned} \frac{d}{dt}x &= y \\ \frac{d}{dt}y &= cy + x - x^2 + xy \end{aligned}$$

with the parameter $c = -0.86....$



Reversed situations of "supercritical" bifurcations (discussed above) are often called subcritical.

- A stable critical point can collide with two other hyperbolic critical points and become unstable. This is called **subcritical pitch-fork bifurcation**. An example is $\frac{d}{dt}x = cx + x^3, \frac{d}{dt}y = -y$. This example is often associated to catastrophe like in the example $\frac{d}{dt}x = cx + x^3 - x^5$, $\frac{d}{dt}y = -y$
- An unstable limit cycle collapses to a stable critical point and becomes an unstable critical point. This is called a **subcritical Hopf bifurcation**.

These situations lead to "catastrophes". The stable equilibrium or cycle "jumps" discontinuously.