8. homework set	**Math118, O.Knill**

8.1 Define a subshift X of finite type over the alphabet $A = \{a, b, c\}$ by forbidding the words aa, ab, cb.

a) Find a finite set of words, with which you can build any sequence in that subshift.

b) Draw the graph which has as vertices the list of words in a) and as directed edges the possible transitions between these words.

c) Write down a few of words in the **language** of this shift.

8.2 The **golden ratio subshift** is the subshift of finite type over the alphabet $\{0, 1\}$ for which the single word 11 is forbidden.

a) Find a list of words, with which you can build any sequence (words of length 1 are allowed too).

b) Find and draw the graph and the adjacency matrix A which is defined as $A_{ij} = 1$ if the word $w_i w_j$ is allowed in the sequence.

c) The **entropy** of the subshift of finite type is the defined as $\log(\lambda)$, where λ is the largest eigenvalue of A. Compute the entropy of the golden ratio subshift. Relate it to the entropy of the full shift and the entropy of the shift for which both the words 11 and 00 are forbidden.

Remark. The entropy of a shift is a measure on how much information is in a sequence shift. A shift with low entropy can be compressed well.

8.3 A map T on the interval $[0, 1]$ is said to **preserve the measure** dx if $\int f(x)\ dx = \int f(T(x)))\ dx$ for any continuous function f. The triple (X, T, dx) is called a **measure preserving dynamical system**.

a) Show that $T(x) = 2x \bmod 1$ preserves the measure dx. In other words, show that (X, T, dx) is a measure preserving dynamical system.

b) Verify that $T(x) = x + \alpha \bmod 1$ preserves the measure dx so that (X, S, dx) is a measure preserving dynamical system.

c) If $A = [a, b]$ is an interval in $[0, 1]$, and T is a measure-preserving system, verify that there exist arbitrary large n such that $T^n(A) \cap A$ has some intersection.

8.4 A function X on $([0, 1], dx)$ is also called a random variable. The **expectation** of X is defined as

$$E[X] = \int_0^1 f(x)\ dx$$

Together with a measure preserving dynamical system, we get a sequence of random variables $X_k(x) = X(T^k(x))$. Two random variables are called **uncorrelated**, if $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$.

a) Verify that for $X(x) = \sin(2\pi x)$ and the dynamical system $T(x) = 2x$, the random variables X and $X(T^k)$ are all uncorrelated.

b) Verify that for $X(x) = \sin(2\pi x)$ and any of the dynamical system $T(x) = x + \alpha$, some pair of random variables X and $X(T^k)$ are correlated.

8.5 We look again at the **cat map** $T(x, y) = (2x + y, x + y)$ on the torus Y which we represent as a square in which opposite sites are identified. Draw the stable and unstable manifolds of the fixed point $(0, 0)$ until the hit themselves. This defines a partition of Y into three sets. Doing the symbolic dynamics gives us a map S from the torus Y to the sequence space X over the alphabet $A = \{a, b, c\}$. This map S defines a conjugation of the cat map T to a subshift (X, σ) of finite type defined by a set of forbidden words of length 2.

a) Find all forbidden words of length 2 of the subshift. To do so, look at the images of the three rectangular sets Y_1, Y_2, Y_3. Hint: You can find three rectangles $T(Y_1), T(Y_2), T(Y_3)$ covering the same space then Y_1, Y_2, Y_3. See the right picture below.

b) Find the graph which belongs to the subshift as well as the adjacency matrix A.

c) Find the entropy of the subshift (the logarithm of the maximal eigenvalue of A) and compare it with the average Lyapunov exponent of the cat map T.