7. Homework set	Math118, O.Knill

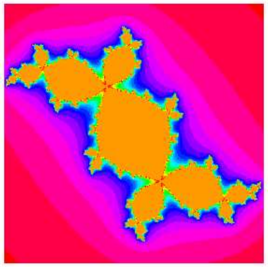
7.1 The Mandelbrot set M can be defined as the set of parameter values c for which $f_c^n(0)$ stays bounded.

a) Show that M is a subset of $|c| \leq 2$.

b) Find and explain a nontrivial symmetry in the Mandelbrot set.

7.2 The **filled in Julia set** K_c are the set of all points z in the comple plane such that $f_c^n(z)$ stays bounded. The Julia set J_c itself is the boundary of that "prisoner set" K_c.

a) Why are all filled in Julia sets of $f_c(z) = z^2 + c$ centrally symmetric? The picture below shows the Douady rabbit.



b) Show that J_c is a compact set.
Hint: Find a radius $r(c)$ such that J_c is contained in $B_r(0) = \{|z| \leq r\}$.

7.3 a) If $T(z) = p(z)$ is a polynomial map, find the number of periodic points of period n, where we count the periodic points with multiplicity and do not require the periodic points to be of minimal period.

b) How many periodic points of minimal period 2 do you expect in general for a cubic map T?

c) Show that every cubic polynomial T can be conjugated to $f_{a,b}(z) = z^3 - 3a^2z + b$ by a linear conjugation.

Hint. If c_1, c_2 are the critical point of T, then conjugate with a translation $S(z) = z + c$ so that the new critical points are centrally symmetric $a, -a$. Then conjugate with $S(z) = dz$ so that the coefficient of z^3 becomes 1.

7.4 De Moivres formula

$$(\cos(nz) + i\sin(nz)) = z^n = (\cos(\theta) + i\sin(\theta))^n$$

shows that $\cos(nz)$ can be written as a polynomial in $\cos(\theta)$. (Just look at the real part of both sides of this identity and use that $\sin^2(\theta) = 1 - \cos^2(\theta)$.

a) Find the Chebychev polynomials $T_1(z), T_2(z), T_3(z)$. What is the Julia set of each of the map $T_k(z)$?

b) Verify that each line segment in the complex plane through 0 which is centrally symmetric is the Julia set of some quadratic polynomial.

7.5 We want to understand why the Julia set of the Newton method applied to $f(z) = z^3 - 1$ must be complicated. Remember that the $T(z) = z - f(z)/f'(z)$ is the Newton method. The Julia set of T is the set of points which are not attracted to one of the fixed points of T. It is the boundary between three regions, the attractors of the fixed points.

a) Verify that the fixed points of T are attractive. Compute the Lyapunov exponent $\lambda(T, x)$, if x is in the basin of attraction of a fixed point.

b) Show that the basins of attractions as well as the Julia set J are T invariant.

c) Verify that the Julia set is invariant under rotation by $2\pi/3$ in the complex plane.