4. Homework set	**Math118, O.Knill**

4.1 a) Prove the following theorem: if $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is a vector field in space which is divergence free $\text{div}(F(x,y,z)) = 0$, then the differential equation $\frac{d}{dt}\vec{x} = F(\vec{x})$ preserves the volume.

Remark. We have done that in two dimensions. You may be able to verify the following formula using Gauss theorem: if D is a region in space then $d/dt \text{vol}(D) = \int\int\int_D \text{div}(F(x,y,z))\, dxdydz$.

b) Under which conditions on a, b, c does the famous ABC system

$$\begin{aligned}\dot{x} &= a\sin(z) + c\cos(y)\\ \dot{y} &= b\sin(x) + a\cos(z)\\ \dot{z} &= c\sin(y) + b\cos(x)\end{aligned}$$

preserve volume?

P.S. By the way, this is a system, where one could expect positive Lyapunov exponent on a substantial subset of torus. But nobody knows how to estimate this.

4.2 Analyze the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\ .\end{aligned}$$

for $\sigma = 0$. What can you say about the equilibrium points for the two-dimensional system if $x = s$ is fixed?

4.3 Reed section 1 and 2 in the booklet "Chaotic evolution and strange attractors" and write down, what Ruelle's view of "turbulence" and "chaos" is. (One small paragraph is enough).

4.4 Verify that the Lorenz system can not have **quasi-periodic solutions**. These are solutions which do not close and which cover a two dimensional torus densely.

4.5 Compute the fractal dimension of the Menger sponge. This three dimensional set can be obtained iteratively as the Cantor set by successively taking away the middle rectangular columns of each complete cube. You see the first two steps of the construction in the picture.