10. homework set	**Math118, O.Knill**

10.1 Consider the 3-body problem in space with interaction potential $V(x) = ||x||^2/2$ and where particles have mass m. Find explicit solution formulas to this problem.

Hint. Go into a coordinate system in which the center of mass is fixed.

Remark. The natural Newton potential depends on the space. The Harmonic oscillator potential $||x||^2$ can be considered the natural 0 dimensional Newton potential

dimension	potential	force
$0D$ Euclidean	$V(x) = \|\|x\|\|^2$	$F(x) = -2x$
$1D$ Euclidean	$V(x) = \|\|x\|\|$	$F(x) = -x/\|\|x\|\|$.
$2D$ Euclidean	$V(x) = \log \|\|x\|\|$	$F(x) = -x/\|\|x\|\|^2$.
$3D$ Euclidean	$V(x) = 1/\|\|x\|\|$	$F(x) = -x/\|\|x\|\|^3$.
$4D$ Euclidean	$V(x) = 1/\|\|x\|\|^2$	$F(x) = -2x/\|\|x\|\|^4$.

In general one can compute the natural Newton potential by looking at the solutions of the Poisson equation $\Delta V = \delta_0$ which gives the solution V. In Euclidean space as well as tori, or sphere, this can be solved with Fourier theory.

10.2 The **antropic principle** is a cheap but effective philosophical explanation for many things. It was introduced in 1973 by the theoretical physicist **Brandon Carter** and has been discussed in popularized in the bestseller of **Steven Hawkings** "A short history of time". The strong antropic principle answers the question, why a physical law or physical fact holds by demonstrating that if the law would be violated, then human life would be impossible: no human person (antropos) could observe it. The principle can be used for example to explain why energy conservation is a reasonable physical law: without it, spontaneous runaway processes could produce an unbounded amount of energy, destroying everything near it. Use the proof of the first Kepler law to verify that bounded planetary motion in four dimensional space is exceptional and argue whether the antropic principle excludes a universe with four dimensional Euclidean space (five dimensional space time).

Remark: You can assume that in d-dimensional space, the natural Newton force is $-Gm_im_j\vec{r}/||x||^d$, where $\vec{r}$ is the vector between the two bodies. There are physical theories called Kaluza-Klein theories which propose higher dimensional space but this is no more in the realm of classical mechanics.

10.3 We consider in this problem set an n body problem, where particles interact only with their neighbors. We look at the **Toda system** which is a famous n body problem which is **integrable** and exhibits fancy solutions called **solitons**. The system is a discretization of the Korteweg de Vries equation (KdV) $u_t = 6uu_x - u_{xxx}$.

One can visualize the particles located on a chain. The potential energy has the form

$$V(q) = \sum_i f(q_i - q_{i-1}) .$$

Consider a chain of particles $q_n = q_{n+N}$ with mass $m_i = 1$ and with potential $f(q) = e^{-q}$. The motion of these particles is given by the differential equations

$$\frac{d^2}{dt^2} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}} ,$$

Verify that after a coordinate transformation

$$4a_n^2 = e^{q_{n+1}-q},\ 2b_n = p_n ,$$

the equations

$$\begin{aligned} \dot{q}_n &= p_n , \\ \dot{p}_n &= e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}} \end{aligned}$$

go into

$$\begin{aligned} \dot{a}_n &= a_n(b_{n+1} - b_n) \\ \dot{b}_n &= 2(a_n^2 - a_{n-1}^2) . \end{aligned}$$

10.4 Given a_n, b_n, define the matrices

$$L = \begin{bmatrix} b_1 & a_1 & 0 & \cdot & 0 & a_N \\ a_1 & b_2 & a_2 & \cdot & \cdot & 0 \\ 0 & a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{N-2} & 0 \\ 0 & \cdot & \cdot & a_{N-2} & b_{N-1} & a_{N-1} \\ a_N & 0 & \cdot & 0 & a_{N-1} & b_N \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & a_1 & 0 & \cdot & 0 & -a_N \\ -a_1 & 0 & a_2 & \cdot & \cdot & 0 \\ 0 & -a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{N-2} & 0 \\ 0 & \cdot & \cdot & -a_{N-2} & 0 & a_{N-1} \\ a_N & 0 & \cdot & 0 & -a_{N-1} & 0 \end{bmatrix} .$$

A matrix L is called a Jacobi matrix. Verify that the Toda system is equivalent to the **Lax equations**

$$\dot{L} = [B, L] = BL - LB .$$

10.5 We show that for a Lax equations $\dot{L} = [B, L]$ with $B^T = -B$, the eigenvalues of L are **integrals of motion**.

a) Consider the differential equation $\dot{S} = BS$ with $S(0) = 1 = I_n$ in the space of matrices. Show that $SS^T = I_n$ for all times.
Hint: $\dot{S}^T = S^T B^T = -S^T B$.

b) Verify the formula

$$L(0) = S(t)^T L(t) S(t)$$

by verifying that $d/dt(S^T LS) = 0$.

c) Conclude that the eigenvalues of L are preserved.

d) If the eigenvalues are preserved, then also the trace of L, the sum of the eigenvalues is perserved. What is the physical meaning of this integral?

e) If the eigenvalues of L are preserved, then also the trace of L^2 is preserved. What is the physical meaning of this integral?