Homework 1. Week	Math118, O.Knill

1.1 We consider the interval map $f(x) = f_4(x) = 4x(1-x)$. For this particular value, the logistic map is also called the **Ulam map**. We have met it in the first lecture, when we saw that a computer does not "know" the distributative law.

a) Find all the fixed points of the map $T(x) = f(x)$.

b) Analyze the stability of these fixed points. For each point, just tell, whether it is stable or unstable.

c) Draw a graph of this map and start iterating the map using the cobweb construction with the initial value 0.3. Do at least 5 iterates.

1.2 Consider the map $Q_c(x) = x^2 + c$. It is called the **quadratic map**. Again, c is a constant parameter.

a) Verify that this map undergoes a **saddle node bifurcation** (which is also called **blue-sky bifurcation** because of obvious reasons, periodic appear or disappear out of the blue sky. It is also called **tangent bifurcation**). For which value of c does this happen?

b) Analyze the stability of the periodic orbits near the bifurcation value.

c) What happens with the orbits for parameter values c for which we have no fixed point?

1.3 a) Look at the fixed points of the map $Q_c(x) = x^2 + c$ for $c = -0.5$ and determine their stability.

b) Look at the fixed point of the map for $c = -1$ and determine its stability.

c) Verify that the map undergoes a **flip bifurcation** at the parameter $c = -3/4$.

1.4 We define the map $f(x) = 5x + \sin(\pi x)$ mod 1 on the interval $[0, 1]$.

Sideremark: Because after identifying 0 and 1, the interval closes to a circle, the map can be considered a smooth map on the circle. f is an example of a **circle map**.

a) What is the Lyapunov exponent of the orbit of the map f wich an initial condition $x_0 = 1/2$?

b) Verify that the Lyapunov exponent of every orbit of f is positive.

1.5 We have shown that the **Ulam map** $f_4(x) = 4x(1-x)$ is conjugated to the tent map $T(x) = 1 - 2|x - 1/2|$

a) Draw the graph of the the iterates $T^2(x), T^3(x)$ of the tent map. Use the fact that the tent map is piecewise linear.

b) Use the conjugation result to sketch the graphs of the second iterate $f_4^2(x)$ and third iterate $f_4^3(x)$ of the Ulam map.

c) Conclude that f_4^n has 2^n fixed points and therefore, that the Ulam map f has 2^n periodic points of period n.

d) What is the Lyapunov exponent of a periodic point of period n?

2. Homework set — Math118, O.Knill

2.1 a) Realize the Henon map

$$T(x,y) = (y + 1 - ax^2, bx)$$

as a second order difference equation. A second order difference equation is a recursion of the form $x_{n+1} = F(x_n, x_{n-1})$.

b) An orbit $x_0, x_1, x_2, ...$ of a difference equation is called periodic, if there exists an integer n such that $x_{k+n} = x_k$ for all k.
Verify that periodic points of the difference equation define periodic points of T.

c) Find a periodic point of prime period 2 of the Henon map in the case $a = 1, b = 1$. The map is then

$$T(x,y) = (1 - x^2 + y, x)$$

. The notion **prime period** 2 means that it should not be a fixed point.

d) (This is optional. Do it only if you have time and access to a CAS.) Can you find formulas for period 2 orbits for general a, b. You might need a computer algebra system. If you use a computer algebra system, find also all periodic orbits of the Henon map with prime period 3 and 4. The formulas can get messy.

2.2 a) Analyze the stability and nature of all the fixed points of the cubic Henon map $T(x,y) = (cx - x^3 - y, x)$ depending on the parameter c.

b) Find the **bifurcation points**, which are parameter values, where the stability of one of these fixed points changes.

2.3 Consider the map

$$T(x,y) = (2x + 3y, x + 2y)$$

on the torus.

a) Is T is area preserving?

b) Verify that the fixed point $(0,0)$ of T is hyperbolic. What are the stable and unstable manifolds of this fixed point?

c) Find the Lyapunov exponents of each orbit of T as well as the entropy, which is the average of the Lyapunov exponent.

d) Argue, why $T(x,y) = (2x + 3y + \epsilon \sin(x), x + 2y)$ has homoclinic points for small ϵ.

Remark. No formal proof is required in d). Just explain in words.

2.4 a) Compute the Lyapunov exponent of fixed points of the Henon map $T(x,y) = (1 - x^2 + y, x)$.

b) Compute the Lyapunov exponent of the periodic orbit you found in [2.1c].

c) What is the Lyapunov exponent of an initial point on the stable manifold to the periodic point you found in b)?

Remark. No computation is necessary in c).

2.5 a) Prove that the cat map $T(x,y) = (2x + y, x + y)$ on the torus is not integrable.

b) Show that the cat map $T(x,y) = (2x + y, x + y)$ defined on the plane is integrable.

Remark: It is possible to give an explicit integral for T but it is also possible just give arguments.

3. Homework set — Math118, O.Knill

3.1 a) Is the flow generated by the differential equation

$$\frac{d}{dt}x = -2\sin(x+2y) - x\sin(xy), \; \frac{d}{dt}y = \sin(x+2y) + y\sin(xy)$$

area-preserving?

b) Find a function $H(x,y)$, such that the differential equation can be rewritten as

$$\frac{d}{dt}x = H_y(x,y), \; \frac{d}{dt}y = -H_x(x,y)$$

c) Is the system integrable?

3.2 a) Assume that $\dot{x} = F(x)$ is a differential equation defined in an annulus $A = \{1 < (x^2+y^2) < 4\}$ and assume that A is left invariant under the differential equation. Assume that $\mathrm{div}(F)(x,y) < 0$ everywhere in the annulus. Prove that there can exist maximally one cycle in A.

b) Assume that $\dot{x} = F(x)$ is a differential equation defined in the disk $D = \{x^2+y^2 < 1\}$. Assume that this disk is left invariant under the differential equation. Assume that $\mathrm{div}(F) < 0$ everywhere in the disk. Prove that there can not exist any limit cycle in D.

3.3 a) Verify **Dulacs criterion**: assume $\dot{x} = F(x)$ is a differential equation in a region D of the plane. If there exists a smooth function g, such that $\mathrm{div}(gF(x))$ has no zeros in D, then there are no closed cycles in D.

b) Use Dulacs criterion to show that

$$\begin{aligned}\frac{d}{dt}x &= x(2-x-y)\\ \frac{d}{dt}y &= y(4x-x^2-3)\end{aligned}$$

has no closed cycles in the region $D = \{x > 0, y > 0\}$.

Hint. This is hard to guess: try $g(x,y) = 1/(xy)$.

3.4 a) The **clycolytic oscillator** is a model for the biochemical process **glycloysis**:

$$\begin{aligned}\frac{d}{dt}x &= -x + ay + x^2y\\ \frac{d}{dt}y &= b - ay - x^2y\end{aligned}$$

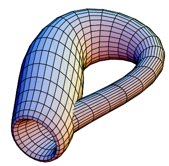
The system depends on two parameters $a > 0, b > 0$. The variable x is the concentration of ADP (adenosine diphosphate) and y is the concentration of F6P (fructose-6-phosphate). The parameter space is divided into two regions. One region, where the fixed point $(b, b/(a+b^2)$ is stable, the other, where the fixed point is unstable and where a stable limit cycle exists. Find the boundary between these two regions. When passsing this boundary, Hopf bifurcations occur.

b) Verify that the differential equation

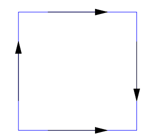
$$\begin{aligned}\frac{d}{dt}x &= y - (x^{11} - 100x)\\ \frac{d}{dt}y &= -x\ .\end{aligned}$$

has a unique limit cycle.

3.5 The **Klein bottle** is an example of a two-dimensional surface. It can not be realized without selfintersection in space. Explore whether the Poincare-Bendixon theorem holds on the Klein bottle or not.



Hint: you can build the Klein bottle as a square at which left and right are identified in the opposite orientation and top and bottom are identified with the same orientation. Start by gloing the top and bottom together. This gives a cylinder. Then, instead of glueing the cylinder together at the end (which produces a torus), glue them together in opposite direction.



3.6* (These are unsolved problems and therefore optional).
a) (**Dulac problem**) Verify that a differential equation

$$\begin{aligned}\frac{d}{dt}x &= p(x,y)\\ \frac{d}{dt}y &= q(x,y)\end{aligned}$$

with polynomials p and q of degree n has only finitely many limit cycles. Find a bound for their number $H(n)$.

b) (Special case of **Hilberts 16'th problem**) Show that a Lienard system with $g(x) = x$ and polynomial $F(x)$ of degree $2k+1$ has at most k limit cycles.

4.1 a) Prove the following theorem: if $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z)$ is a vector field in space which is divergence free $\mathrm{div}(F(x,y,z)) = 0$, then the differential equation $\frac{d}{dt}\vec{x} = F(\vec{x})$ preserves the volume.

Remark. We have done that in two dimensions. You may be able to verify the following formula using Gauss theorem: if D is a region in space then $d/dt\mathrm{vol}(D) = \int\int\int_D \mathrm{div}(F(x,y,z))\, dxdydz$.

b) Under which conditions on a, b, c does the famous ABC system

$$\begin{aligned} \dot{x} &= a\sin(z) + c\cos(y) \\ \dot{y} &= b\sin(x) + a\cos(z) \\ \dot{z} &= c\sin(y) + b\cos(x) \end{aligned}$$

preserve volume?

P.S. By the way, this is a system, where one could expect positive Lyapunov exponent on a substantial subset of torus. But nobody knows how to estimate this.

4.2 Analyze the Lorenz system

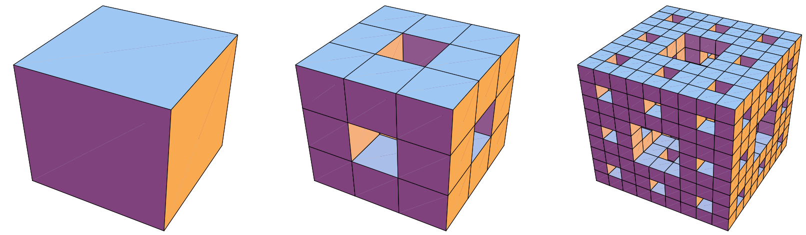
$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\ . \end{aligned}$$

for $\sigma = 0$. What can you say about the equilibrium points for the two-dimensional system if $x = s$ is fixed?

4.3 Reed section 1 and 2 in the booklet "Chaotic evolution and strange attractors" and write down, what Ruelle's view of "turbulence" and "chaos" is. (One small paragraph is enough).

4.4 Verify that the Lorenz system can not have **quasi-periodic solutions**. These are solutions which do not close and which cover a two dimensional torus densely.

4.5 Compute the fractal dimension of the Menger sponge. This three dimensional set can be obtained iteratively as the Cantor set by successively taking away the middle rectangular columns of each complete cube. You see the first two steps of the construction in the picture.



5.1 Given a rectangle of length 1 and height $b > 1$. We play billiards in this table. For which angles θ does a trajectory (which does not hit a corner) get arbitrarily close to any point on the boundary of the table?

5.2 We have seen that for every period n with prime n, there is a periodic orbit of a billiard. We have done this by by maximizing the length functional $H(x_1, ..., x_n)$, which is the total length of the closed trajectory. We have assumed that the integer n has no nontrivial factor, because we did not want to have a periodic orbit of some smaller period.

a) Can you prove that the result actually holds for any n? There is a periodic orbit of minimal period n.

b) Prove that there are at least two periodic orbits of period 5 in the table $x^4 + y^4 \leq 1$.

5.3 Here is an application of the Kronecker dynamical system $x \to x + \alpha$. Consider the first digits of the powers

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ..$$

Can you determine how often each digit occurs in average? Which digit does occur more often, the digit 8 or the digit 9?

Hint. If a power 2^k starts with the digit 5 then $5 \cdot 10^m \leq 2^k < 6 \cdot 10^m$ for some m. Take logarithms to the base 10 of this equation and watch out for a Kronecker system. You are allowed to use **Weyls theorem** without proof, which assures that for irrational α, the frequency with which $[k\alpha]$ is in some interval $[a, b]$ is equal to $b - a$. We will prove that later.

5.4 a) A table is called convex, if the line segment connecting two arbitrary points in the table is inside the table. Verify that the billiard map can not be continuous on the annulus $(R/Z) \times [-1, 1]$, if the table is not convex.

b) Verify that the billiard in a half ellipse $x^2/a^2 + y^2/b^2 \leq 1, y \geq 0$ is integrable.

5.5 The string construction allows to construct a table, with a given caustic.

a) Draw a family of tables, which have a regular triangle as a caustic.

b) Draw a family of tables, which have a regular square as caustics.

c) Is there a convex billiard table which has two different caustics, where each caustic is a polygon?

6. Homework set **Math118, O.Knill**

6.1 Let $X = \{0,1\}^Z$ with the distance $d(x, y) = 1/(n+1)$ n is the smallest index such that $x_n \neq y_n$. that is $x_k = y_k$ for $|k| \leq n$ and $x_n \neq y_n$ or $x_{-n} = y_{-n}$.

a) To verify that X is a metric space, have to verify $d(x, y) = d(y, x)$, $d(x, x) = 0$ and the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Hint: Actually a stronger inequality holds: $d(x, z) \leq \max(d(x, y), d(y, z))$. If x, y agree on some interval and y, z agree on an other say bigger interval, then x, z agree at least on the smaller interval.

b) Verify that X is compact: every sequence $x(n)$ in X has an accumulation point. You have to show that there is a subsequence $x(k_l)$ such that $x(k_l)$ converges in X for $l \to \infty$.

Hint. You have to construct an accumulation point using a diagonal type argument (unlike in Cantors case, you do not need to change the diagonal).

6.2 Model an epidemic as a CA. You can use your own model. Otherwise, a suggestion would be that people can either be healthy, sick and non-contagious or sick and contagious, The disease is quite severe so that sick people close to a contagious sick person will become sick to. After incubation, they become themselves contagious, then sick and non-contageous, then sick and contagious and then recover again. Describe the CA rule $\phi(a, b, c)$ which gives the state of the person depending on its own state b, and the states a, c of two neighbors. Can you invent a numbering which includes all CA of the type you consider, analogue to the Wolfram numbering over the alphabet $\{0, 1\}$?

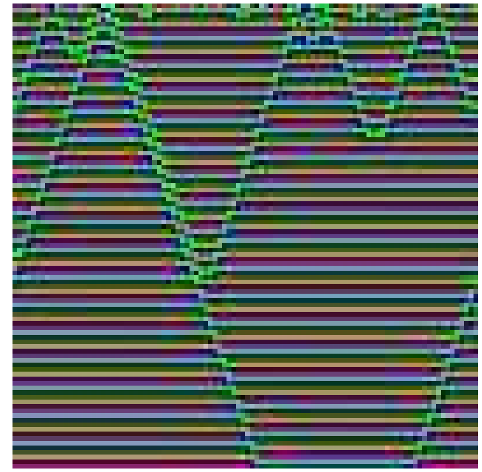
Optional: simulate your CA on the computer.

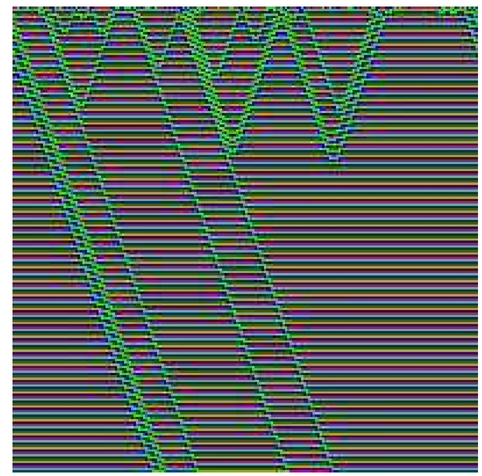
6.3 a) Construct a CA with radius 1 over the alphabet $\{0, 1, 2, 3, 4, 5\}$ for which the speed c satisfies is $c/R = c = 2/5$.

Remark. If you would generalize your construction to any alphabet, you could proceed to prove that the set c/R of speed/radius ratios of one-dimensional cellular automata is dense in the interval $[0, 1]$. Since one can simulate any automaton with alphabet A with an automaton with alphabet $\{0, 1\}$ but with an adjustment of the radius, one can see that also in general, the possible speed ratios c/R are dense in the interval $[0, 1]$.

6.4 a) Verify that the CA T over the alphabet $A = \{0, 1, 2\}$ defined by $\phi(x, y, z) = xyz \bmod 3$ has an attractor $K = \bigcap_n T^n(X)$ on which the the map is isomorphic to an elementary CA. (An elementary CA is a CA in one dimensions over the alphabet $\{0, 1\}$).

b) (optional) The CA T over the alphabet $A = \{0, 1, 2\}$ defined by $\phi(x, y, z) = xyz + 1 \bmod 3$ shows "particles". If two particles interact, they annihilate. Can you verify this picture?





6.5 Invent a CA in two dimensions of your choice. Either look at a simple one which you can analyze completely. Or invent one which models some situation (life, forest fire, spread of passion for dynamical systems etc.) It is possible to do experiments on a computer if you wish but this is not necessary. Coming up with an interesting suggestion is also good.

7. Homework set	**Math118, O.Knill**

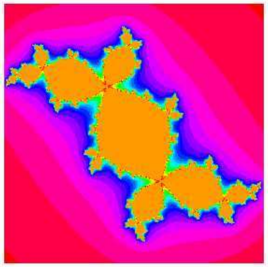
7.1 The Mandelbrot set M can be defined as the set of parameter values c for which $f_c^n(0)$ stays bounded.

a) Show that M is a subset of $|c| \leq 2$.

b) Find and explain a nontrivial symmetry in the Mandelbrot set.

7.2 The **filled in Julia set** K_c are the set of all points z in the comple plane such that $f_c^n(z)$ stays bounded. The Julia set J_c itself is the boundary of that "prisoner set" K_c.

a) Why are all filled in Julia sets of $f_c(z) = z^2 + c$ centrally symmetric? The picture below shows the Douady rabbit.



b) Show that J_c is a compact set.
Hint: Find a radius $r(c)$ such that J_c is contained in $B_r(0) = \{|z| \leq r\}$.

7.3 a) If $T(z) = p(z)$ is a polynomial map, find the number of periodic points of period n, where we count the periodic points with multiplicity and do not require the periodic points to be of minimal period.

b) How many periodic points of minimal period 2 do you expect in general for a cubic map T?

c) Show that every cubic polynomial T can be conjugated to $f_{a,b}(z) = z^3 - 3a^2z + b$ by a linear conjugation.

Hint. If c_1, c_2 are the critical point of T, then conjugate with a translation $S(z) = z + c$ so that the new critical points are centrally symmetric $a, -a$. Then conjugate with $S(z) = dz$ so that the coefficient of z^3 becomes 1.

7.4 De Moivres formula

$$(\cos(nz) + i\sin(nz)) = z^n = (\cos(\theta) + i\sin(\theta))^n$$

shows that $\cos(nz)$ can be written as a polynomial in $\cos(\theta)$. (Just look at the real part of both sides of this identity and use that $\sin^2(\theta) = 1 - \cos^2(\theta)$.

a) Find the Chebychev polynomials $T_1(z), T_2(z), T_3(z)$. What is the Julia set of each of the map $T_k(z)$?

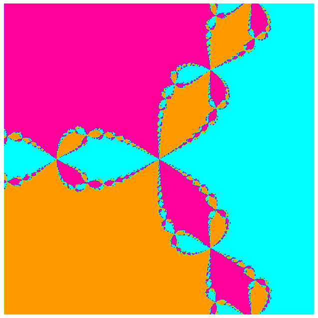
b) Verify that each line segment in the complex plane through 0 which is centrally symmetric is the Julia set of some quadratic polynomial.

7.5 We want to understand why the Julia set of the Newton method applied to $f(z) = z^3 - 1$ must be complicated. Remember that the $T(z) = z - f(z)/f'(z)$ is the Newton method. The Julia set of T is the set of points which are not attracted to one of the fixed points of T. It is the boundary between three regions, the attractors of the fixed points.

a) Verify that the fixed points of T are attractive. Compute the Lyapunov exponent $\lambda(T, x)$, if x is in the basin of attraction of a fixed point.

b) Show that the basins of attractions as well as the Julia set J are T invariant.

c) Verify that the Julia set is invariant under rotation by $2\pi/3$ in the complex plane.



8. homework set	**Math118, O.Knill**

8.1 Define a subshift X of finite type over the alphabet $A = \{a, b, c\}$ by forbidding the words aa, ab, cb.

a) Find a finite set of words, with which you can build any sequence in that subshift.

b) Draw the graph which has as vertices the list of words in a) and as directed edges the possible transitions between these words.

c) Write down a few of words in the **language** of this shift.

8.2 The **golden ratio subshift** is the subshift of finite type over the alphabet $\{0, 1\}$ for which the single word 11 is forbidden.

a) Find a list of words, with which you can build any sequence (words of length 1 are allowed too).

b) Find and draw the graph and the adjacency matrix A which is defined as $A_{ij} = 1$ if the word $w_i w_j$ is allowed in the sequence.

c) The **entropy** of the subshift of finite type is the defined as $\log(\lambda)$, where λ is the largest eigenvalue of A. Compute the entropy of the golden ratio subshift. Relate it to the entropy of the full shift and the entropy of the shift for which both the words 11 and 00 are forbidden.

Remark. The entropy of a shift is a measure on how much information is in a sequence shift. A shift with low entropy can be compressed well.

8.3 A map T on the interval $[0, 1]$ is said to **preserve the measure** dx if $\int f(x)\ dx = \int f(T(x))\ dx$ for any continuous function f. The triple (X, T, dx) is called a **measure preserving dynamical system**.

a) Show that $T(x) = 2x \bmod 1$ preserves the measure dx. In other words, show that (X, T, dx) is a measure preserving dynamical system.

b) Verify that $T(x) = x + \alpha \bmod 1$ preserves the measure dx so that (X, S, dx) is a measure preserving dynamical system.

c) If $A = [a, b]$ is an interval in $[0, 1]$, and T is a measure-preserving system, verify that there exist arbitrary large n such that $T^n(A) \cap A$ has some intersection.

8.4 A function X on $([0, 1], dx)$ is also called a random variable. The **expectation** of X is defined as

$$E[X] = \int_0^1 f(x)\ dx$$

Together with a measure preserving dynamical system, we get a sequence of random variables $X_k(x) = X(T^k(x))$. Two random variables are called **uncorrelated**, if $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$.

a) Verify that for $X(x) = \sin(2\pi x)$ and the dynamical system $T(x) = 2x$, the random variables X and $X(T^k)$ are all uncorrelated.

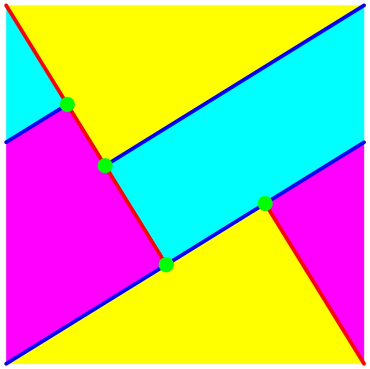
b) Verify that for $X(x) = \sin(2\pi x)$ and any of the dynamical system $T(x) = x + \alpha$, some pair of random variables X and $X(T^k)$ are correlated.

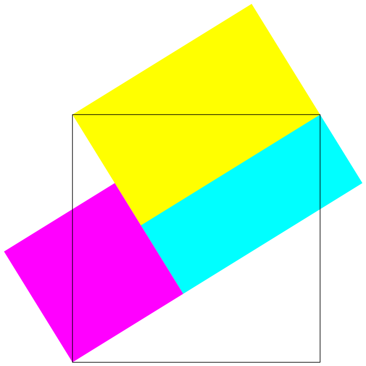
8.5 We look again at the **cat map** $T(x, y) = (2x + y, x + y)$ on the torus Y which we represent as a square in which opposite sites are identified. Draw the stable and unstable manifolds of the fixed point $(0, 0)$ until the hit themselves. This defines a partition of Y into three sets. Doing the symbolic dynamics gives us a map S from the torus Y to the sequence space X over the alphabet $A = \{a, b, c\}$. This map S defines a conjugation of the cat map T to a subshift (X, σ) of finite type defined by a set of forbidden words of length 2.

a) Find all forbidden words of length 2 of the subshift. To do so, look at the images of the three rectangular sets Y_1, Y_2, Y_3. Hint: You can find three rectangles $T(Y_1), T(Y_2), T(Y_3)$ covering the same space then Y_1, Y_2, Y_3. See the right picture below.

b) Find the graph which belongs to the subshift as well as the adjacency matrix A.

c) Find the entropy of the subshift (the logarithm of the maximal eigenvalue of A) and compare it with the average Lyapunov exponent of the cat map T.





9. homework set	**Math118, O.Knill**

9.1 Given a large number $n = pq$, where p, q are prime numbers, consider the "quadratic map" $T(x) = x^2 + c$ modulo n, where c is an integer. The **Pollard rho method** to factor n looks at the orbit of a point x. Since it will eventually be periodic modulo q, we have $x_n = x_k \bmod q$ which means that $x_n - x_k$ has a common factor with n. Find the orbit structure of the dynamical system $T(x) = x^2 + 1 \bmod n$. Because we have a finite set, every point is eventually periodic. This is the reason for the name ρ. An initial point will eventually be caught in a loop. Find all the periods in the case $n = 15$.

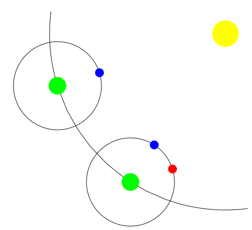
Remark: Assuming the sequence x_n to be random, we need to take about $\sqrt{q}$ iterates to find a factor with probability 1/2. If you have q different objects and you chose k objects, the probability that two are not the same is $q(q-1)...(q-k+1)/q^k = \frac{q!}{(q-k)!q^k}$. That these probabilities are relatively small is called the **Birthday paradox**. If you have a room of 23 people, then then the probability that two have the same birthday is $1 - 365!/(342!365^23) = 0.5072$. Note that $\sqrt{365} = 19.11...$ is close to 23.

9.2 a) Find the continued fraction expansion of the **golden ratio** $(\sqrt{5}+1)/2$ and relate the periodic approximations p_n/q_n with the Fibonacci sequence.

b) Find the continued fraction expansion of the **silver ratio** $1 + \sqrt{2}$. Can you find the rule, which generates p_n and q_n in the the partial fractions p_n/q_n for the silver ratio?

9.3 A **synodic month** is defined as the period of time between two new moons. It is $\alpha = 29.530588853$ days. The **draconic month** is the period of time of the moon to return to the same node. It is $\beta = 27.212220817$ days. Intersections between the path of the moon and the sun are called **ascending and descending nodes**.

Such an intersection is called a **solar eclipse**. In one approximation, it appears in a period of a bit more then 18 years = 6580 days which is called one **Saros cycle**. This cycle and others are obtained from the continued fraction expansion of α/β. It is said that Thales used the Saros cycle cycle to predict the solar eclipse of 585 B.C. The next big eclipse will happen May 26, 2021. Explain at least two of the following Eclipse cycles (one of them should be the soros cycle) with the continued fraction expansion.



cycle	eclipse	synodic	draconic
fortnight	14.77	0.5	0.543
month	29.53	1	1.085
semester	177.18	6	6.511
lunar year	354.37	12	13.022
octon	1387.94	47	51.004
tritos	3986.63	135	146.501
saros	6585.32	223	241.999
Metonic cycle	6939.69	235	255.021
inex	10571.95	358	388.500
exeligmos	19755.96	669	725.996
Hipparchos	126007.02	4267	4630.531
Babylonian	161177.95	5458	5922.999

If you have no Mathematica installed: turn your browser to

http://sofia.fas.harvard.edu/cgi-bin/sofia

You can get continued fractions by entering something like

ContinuedFraction[Pi, 20].

9.4 We have seen that a parabola $y = p_2(x) = ax^2 + by + c$ defines a dynamical system on the two dimensional torus. This construction goes as follows: $p_1(x) = p_2(x+1) - p_2(x), p_0(x) = p_1(x+1) - p_1(x) = \alpha$ so that $p_1(x+1) = p_1(x) + \alpha$, and $p_2(x+1) = p_2(x) + p_1(x)$. If $x_n = p_1(x+n)$ and $y_n = p_2(x+n)$, then $(x_{n+1}, y_{n+1}) = (x_n + \alpha, y_n + x_n)$.

The curve $f(x) = ax^3 + bx^2 + cx + d$ induces a dynamical system on the three dimensional torus. Find this system and determine whether it preserves volume.

9.5 A widely used data encryption technique goes under the name RSA. The security of this encryption is based on the empirical fact that it is hard to factor large integers $n = pq$. Some of the best methods to factor integers goes back to Fermat: assume we can find a second root y of $x^2 \bmod n$, then $x^2 = y^2 \bmod n$ so that $(x-y)(x+y) = 0 \bmod n$ and $gcd(x-y, n)$ is a factor of n. Finding square roots is difficult directly. The **holy grail** is to find numbers y such that $z = y^2 \ mod\ n$ is so small that one can factor them. Having enough such numbers allows to find small squares using a sieaving technique. The Morison-Brillhard method starts with constructing small integers by doing the continued fraction expansion of $\sqrt{n}$. Explain why the periodic approximation $\sqrt{n} \sim p_n/q_n$ produces numbers p_n for which the square p_n^2 is small modulo n. How big do you expect these numbers to be?

Remark. Want to earn 20'000 Dollars? You can do so by factoring the RSA-640 which has 193 digits:

$n = 3107418240490043721350750035888567930037346022842727545720161948823206$
$4405180815045563468296717232867824379162728380334154710731085019195485290 07$
$33772482278352574238645401469173660247765234660 9.$

See
http://www.rsasecurity.com/rsalabs/node.asp?id=2093
Note that Mathematica has built in factorization techniques. But typing in

FactorInteger[n]

and waiting will most likely will not earn you the prize. Unless you are Indiana Jones ...

10. homework set	**Math118, O.Knill**

10.1 Consider the 3-body problem in space with interaction potential $V(x) = ||x||^2/2$ and where particles have mass m. Find explicit solution formulas to this problem.

Hint. Go into a coordinate system in which the center of mass is fixed.

Remark. The natural Newton potential depends on the space. The Harmonic oscillator potential $||x||^2$ can be considered the natural 0 dimensional Newton potential

dimension	potential	force
$0D$ Euclidean	$V(x) = \|\|x\|\|^2$	$F(x) = -2x$
$1D$ Euclidean	$V(x) = \|\|x\|\|$	$F(x) = -x/\|\|x\|\|$.
$2D$ Euclidean	$V(x) = \log \|\|x\|\|$	$F(x) = -x/\|\|x\|\|^2$.
$3D$ Euclidean	$V(x) = 1/\|\|x\|\|$	$F(x) = -x/\|\|x\|\|^3$.
$4D$ Euclidean	$V(x) = 1/\|\|x\|\|^2$	$F(x) = -2x/\|\|x\|\|^4$.

In general one can compute the natural Newton potential by looking at the solutions of the Poisson equation $\Delta V = \delta_0$ which gives the solution V. In Euclidean space as well as tori, or sphere, this can be solved with Fourier theory.

10.2 The **antropic principle** is a cheap but effective philosophical explanation for many things. It was introduced in 1973 by the theoretical physicist **Brandon Carter** and has been discussed in popularized in the bestseller of **Steven Hawkings** "A short history of time". The strong antropic principle answers the question, why a physical law or physical fact holds by demonstrating that if the law would be violated, then human life would be impossible: no human person (antropos) could observe it. The principle can be used for example to explain why energy conservation is a reasonable physical law: without it, spontaneous runaway processes could produce an unbounded amount of energy, destroying everything near it. Use the proof of the first Kepler law to verify that bounded planetary motion in four dimensional space is exceptional and argue whether the antropic principle excludes a universe with four dimensional Euclidean space (five dimensional space time).

Remark: You can assume that in d-dimensional space, the natural Newton force is $-Gm_im_j\vec{r}/||x||^d$, where $\vec{r}$ is the vector between the two bodies. There are physical theories called Kaluza-Klein theories which propose higher dimensional space but this is no more in the realm of classical mechanics.

10.3 We consider in this problem set an n body problem, where particles interact only with their neighbors. We look at the **Toda system** which is a famous n body problem which is **integrable** and exhibits fancy solutions called **solitons**. The system is a discretization of the Korteweg de Vries equation (KdV) $u_t = 6uu_x - u_{xxx}$.

One can visualize the particles located on a chain. The potential energy has the form

$$V(q) = \sum_i f(q_i - q_{i-1}) .$$

Consider a chain of particles $q_n = q_{n+N}$ with mass $m_i = 1$ and with potential $f(q) = e^{-q}$. The motion of these particles is given by the differential equations

$$\frac{d^2}{dt^2} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}} ,$$

Verify that after a coordinate transformation

$$4a_n^2 = e^{q_{n+1}-q}, \; 2b_n = p_n ,$$

the equations

$$\begin{aligned} \dot{q}_n &= p_n , \\ \dot{p}_n &= e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}} \end{aligned}$$

go into

$$\begin{aligned} \dot{a}_n &= a_n(b_{n+1} - b_n) \\ \dot{b}_n &= 2(a_n^2 - a_{n-1}^2) . \end{aligned}$$

10.4 Given a_n, b_n, define the matrices

$$L = \begin{bmatrix} b_1 & a_1 & 0 & \cdot & 0 & a_N \\ a_1 & b_2 & a_2 & \cdot & \cdot & 0 \\ 0 & a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{N-2} & 0 \\ 0 & \cdot & \cdot & a_{N-2} & b_{N-1} & a_{N-1} \\ a_N & 0 & \cdot & 0 & a_{N-1} & b_N \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & a_1 & 0 & \cdot & 0 & -a_N \\ -a_1 & 0 & a_2 & \cdot & \cdot & 0 \\ 0 & -a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{N-2} & 0 \\ 0 & \cdot & \cdot & -a_{N-2} & 0 & a_{N-1} \\ a_N & 0 & \cdot & 0 & -a_{N-1} & 0 \end{bmatrix} .$$

A matrix L is called a Jacobi matrix. Verify that the Toda system is equivalent to the **Lax equations**

$$\dot{L} = [B, L] = BL - LB .$$

10.5 We show that for a Lax equations $\dot{L} = [B, L]$ with $B^T = -B$, the eigenvalues of L are **integrals of motion**.

a) Consider the differential equation $\dot{S} = BS$ with $S(0) = 1 = I_n$ in the space of matrices. Show that $SS^T = I_n$ for all times.
Hint: $\dot{S}^T = S^T B^T = -S^T B$.

b) Verify the formula

$$L(0) = S(t)^T L(t) S(t)$$

by verifying that $d/dt(S^T LS) = 0$.

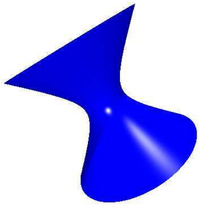
c) Conclude that the eigenvalues of L are preserved.

d) If the eigenvalues are preserved, then also the trace of L, the sum of the eigenvalues is perserved. What is the physical meaning of this integral?

e) If the eigenvalues of L are preserved, then also the trace of L^2 is preserved. What is the physical meaning of this integral?

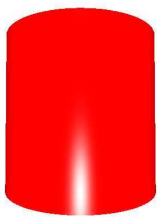
11. homework set	Math118, O.Knill

11.1 Describe the geodesic flow on the one-sheeted hyperboloid. How does a typical geodesic look like. Find one periodic geodesic.

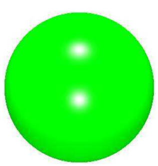


11.2 When you empty a paper towel roll you obtain a cylinder. On this cylinder you find a spiral curve. Prove first that this curve a geodesic. Assume the cylinder is $x^2 + y^2 = 1, 0 \leq z \leq 1$. We play surface billiard: the geodesic curve is reflected at the boundaries $z = 0, z = 1$. Find the return map $T(\theta, \phi) = (\theta_1, \phi_1)$, where $(x, y, z) = (\cos(\theta), \sin(\theta), 0)$ and ϕ is the impact angle.

Optional: From what you know about billiards, can you cut away part of the cylinder to get a chaotic surface billiard?

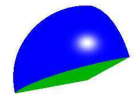


11.3 For a surface of revolution which is symmetric to the xy plane and for which $r(z) \to 0$ for $|z| \to \infty$ we define a Poincare map for the geodesic flow. Sart with a point x on the surface in the xy plane and a unit vector v, follow the geodesic flow along the surface until it comes back to the surface.
This defines return map $T(\theta, \phi)$ on the annulus $(\theta, \phi) \in T \times [0, \pi]$. Describe this map in the case of the sphere.



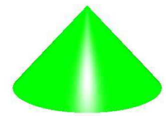
11.4 We play **surface billiard** in the triangular surface obtained by intersecting the unit sphere with the first octant in space. Prove that this billiard is integrable.

Hint. Remember how you analyzed billiards in the rectangle? A similar idea applies here.



11.5 We play **surface billiard** in the half cone $x^2 + y^2 = 1 - z^2$. Prove that there are orbits which never close.

Hint. Similar than for the flat torus or the cylinder one can find geodesics by cutting up the surface and flattening it.



11.6 (optional) Can you prove the statement made in class that on a flat torus, the wave front $K_t(x)$ of a point becomes dense on the torus in the sense, given $\epsilon > 0$, there is a s such that K_t intersects every disc of radius ϵ for $t > s$.

Remark: We do not know whether this statment stays true, if you allow the torus to be bumpy. Nor do we know whether the caustic $C_t(x)$ becomes dense in general. ($C_t(x)$ is the empty set for the flat torus).