CELLULAR AUTOMATA II **Math118, O. Knill**

ABSTRACT. This page contains three mathematical results: the Curtis-Hedlund-Lyndon theorem which says that every continuous, translational invariant map on X is a CA, the proof that σ is chaotic in the sense of Devaney and on a rather technical proof that the topological entropy which we define for CA agrees with the classical topological entropy for general topological dynamical systems.

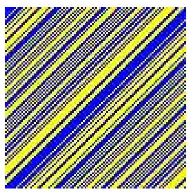
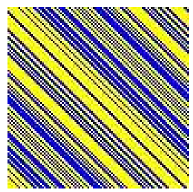
THE CURTIS-HEDLUND-LYNDON THEOREM.

> For every continuous map T on $X = A^Z$ which commutes with σ, there is a finite set $F = \{-R, ..., R\}$ and a map ϕ such that $T(x)_n = \phi(x_{n-R}, ..., x_{n+R})$.

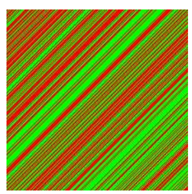
PROOF.
(i) We claim that the map f from X to A defined by $f(x) = T(x)_0$ depends only on $\{x_i, i \in F(x)\ \}$, where $F(x)$ is some finite set.
Proof: If this were not true, there existed a sequence $x(n_k)$ in X with $n_k \to \infty$ such that $x_l = x(n_k)_l$ for $l \neq n_k$ and $x_l \neq x(n_k)_l$ for $l = n_k$ and $T(x(n_k)) \neq T(x)$. Because $x(n_k) \to x$ for $k \to \infty$, the continuity of T implies that $T(x(n_k)) = T(x)$ eventually because of the finiteness of the alphabet. This is a contradiction to $T(x(n_k)) \neq T(x)$ for all k.

(ii) The set $F(x)$ is independent of x.
Proof. First of all, $x \to m(x)$, where $m(x) = \min(F(x))$ and $x \to M(x)$, where $M(x) = \max(F(x))$ are continuous. This implies that $x_n \to x$ implies $F(x_n) = F(x)$ if $d(x_n, x)$ is close enough. The set $F(x)$ is invariant under the shifts σ by assumption. Assume, there exist two points x, y, where $F(x) \neq F(y)$. We can find z and sequence of translations σ^{n_j} such that $\sigma^{n_j}(z) \to y$ and a sequence of translations m_k such that $\sigma^{m_k}(z) \to y$. We have $F(z) = F(\sigma^{n_j} z)$ and $F(z) = F(\sigma^{m_k} z)$ and so $F(x) = F(y)$.

ISOMORPHIC AUTOMATA. Some of the elementary automata are isomorphic. For example, the parity transformation $P(x)_n = x_{-n}$, then $P^{-1}TP$ is a new elementary automaton with a different number. Also $C(x)_k = (1 - x_k)$ which changes 0 and 1 brings a new automata $C^{-1}TC$. Many of the 256 different rules lead to isomorphic systems. Counting the equivalence classes reduces the number 256 to 88. The pictures to the right show rule 170 and rule 240, the left and right shift.

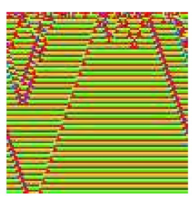
THE "CHAOTIC" SHIFT. The shift map σ is also CA with rule 240.



> CA is chaotic in the sense of Devaney: it has a dense set of periodic points and has a dense orbit.

PROOF. To get a dense orbit, enumerate all finite words w_k and concatenate them together to an infinite sequence y, for $k > 0$. Define $x_k = y_{|k|}$. $T^n(x)$ is dense.
For every x, and every ϵ, there exists a N-periodic sequence y such that $d(x, y) < \epsilon$.

PARTICLES INTERACTIONS. Automata with nearest neighbor interaction and larger alphabets can exhibit already quite interesting behavior. Physicists are intrigued by the similarity to particle physics. Certain configurations travel with some speed, interact and destroy each other like real particles. The picture to the right shows the automaton over the alphabet $Z_p, p = 9$ with $\phi(a, b, c) = a * b * c + 1$. If the CA rule is the "physics" of the "CA micro world", one calls **particles** elements in X which are constant outside some interval and which satisfy $T^n(x) = \sigma^m(x)$. They have speed $v = m/n$. If you are lucky, the interaction of particles produces new particles.



SUBSHIFTS. A closed σ-invariant subset X of A^Z is called a **subshift**. If a subshift X is invariant under a CA map T, we can look at the system (X, T). Examples:

a) If $x = (..., 0, 1, 1, 0, 1, 1, 0, 1, 1, ...)$, then $X = \{x, \sigma(x), \sigma^2(x)\}$ is a subshift. More generally, the set of all M-periodic sequences forms a subshift. Restricting a CA map T onto X means simulating the CA with periodic boundary conditions.

b) Take all sequences with alphabet $\{a, b, c\}$, so that transitions $a- > b- > c- > a$ and $b- > b$ are possible. The space X with words like $(, ..., abcabcabbcabbbcabcabbbbc, ...)$. is an example of a **subshift of finite type**.

c) If T is a cellular automaton map and X is a subshift, then $T(X)$ is a subshift. It is called a **factor** of the original subshift. That is how CA were first introduced by Hedlund.

ATTRACTOR. The image $X_1 = T(X_0)$ of the set of all configurations $X_0 = A^{Z^d}$ is a T invariant subshift. The image $X_2 = T(X_1)$ is invariant too etc. We obtain a nested sequence of subsets $X_0 \supset X_1 \supset X_2 ....$ The limit $X = \bigcap_k X_k$ is called the **attractor** of the cellular automaton. It is a T-invariant subshift.
EXAMPLES. For the shift σ, the attractor is the entire set A^Z. For the rule 0-automaton, the attractor is a single point.

TOPOLOGICAL ENTROPY OF 1D CA. The topological entropy of a 1D CA is defined as

$$h(T) = \lim_{K \to \infty} \lim_{N \to \infty} \frac{\log(R(N, K))}{N} .$$

where $R(N, K)$ be the number of distinct rectangles of width K and height N which occur in a space-time diagram of T.
The picture to the right shows a rectangle $R(N, K)$ for an automaton, where the attractor is a point. Here $R(N, K)$ depends on K but stays bounded in N. The entropy is zero.

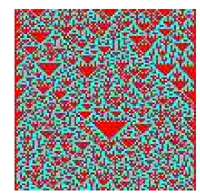
EXAMPLE. The shift $T = \sigma$ has the maximal possible entropy $\log(|A|)$. Take a random sequence x, then $T^n(x)$ will be random sequences too. We have $R(N, K) = |A|^N$.

TOPOLOGICAL ENTROPY IS DIFFICULT TO COMPUTE:

> THEOREM (Hurd,Kari and Culik) Given $\epsilon > 0$. There is no computer algorithm which when given as an input the rule of the CA, the output is the topological entropy up to accuracy ϵ.

The strategy of the proof is to relate the problem of calculating the entropy to the "stopping problem of Turing machines, which is a undecidable problem: there exists no algorithm which takes a Turing machine and decides whether it halts or not.

BOUNDARY CONDITION. If an initial sequence x is periodic, satisfying $x_{i+N} = x_i$ for all i, then $T(x)$ is periodic. We can then watch $x_1, ..., x_N$ and know the entire sequence. In this case, the possible configurations are finite, namely $|A|^N$, where $|A|$ is the cardinality of the alphabet A. The cellular automata map is a map on a finite set X_N.
We can also take fixed boundary conditions, assuming that $x_0 = x_N = 0$. In analogy to PDE's (and CA are in a sense discrete PDE's), one could call this **Dirichlet boundary conditions**.



GROWTH OF LARGEST ATTRACTOR. For a fixed automaton we can look at the size $s(N)$ of the largest attractor on the subshift $X = X_N$ set N periodic sequences. Define the growth rate

$$0 \leq \limsup_N \frac{1}{N} \log(s(N)) \leq \log |A|$$

This growth rate is different from the topological entropy in general: the growth rate of the shift σ is 0, while the topological entropy is $\log |A|$.

GENERAL DEFINITION OF TOPOLOGICAL ENTROPY. The topological entropy of a continuous map T on a compact space X is in general defined as $h(T) = \lim_{\epsilon \to 0} \overline{\lim}_{n \to \infty} \log(M(n, \epsilon))/n$, where $M(n, \epsilon)$ is the minimal number of ϵ-balls in the metric $d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y)$ which cover X.

The topological entropy of the CA agrees with the general topological entropy:

PROOF. Given two $(N, 2K + 1)$-rectangles A, B in the space-time diagram. Enumerate the rows of A and B starting from the bottom with $A_1, \dots, A_N$ and $B_1, \dots, B_N$ and take two elements $x, y \in X$ such that

$$A_j = (T^j(x)_{-K}, \dots T^j(x)_{-1}, T^j(x)_0, T^j(x)_1, \dots, T^j(x)_K) ,$$

$$B_j = (T^j(y)_{-K}, \dots T^j(y)_{-1}, T^j(y)_0, T^j(y)_1, \dots, T^j(y)_K) .$$

Because $A_j = B_j$ if and only if $d(T^j(x), T^j(y)) < 2^{-K}$, we know that $A = B$ implies $d_N(x, y) < 2^{-K}$. On the other hand, if $x, y \in X$ satisfy $d_N(x, y) \geq 2^{-K}$, we have two different rectangles. With

$$M(N, 4 \cdot 2^{-k}) \leq R(N, 2K + 1) \leq M(N, 2^{-k}/4) .$$

(i) Left inequality. Take for each $R(N, 2K + 1)$ rectangles A a point x such that

$$A_1 = (x_{-K}, \dots x_{-1}, x_0, x_1, \dots, x_K) .$$

This gives a finite set $Y \subset X$ with $R(N, 2K + 1)$ points. Every point $x \in X$ has distance $\leq 2 \cdot 2^{-K}$ to one of the points in Y. The $R(N, 2K + 1)$ balls of radius $4 \cdot 2^{-k}$ with midpoints in Y cover X. This proves a).

(ii) Right inequality: two different points in Y have distance $\geq 2^{-K}/2$. We need therefore at least $R(N, 2K+1)$ balls of radius $2^{-K}/4$ to cover X.

The two inequalities together give $R(N, 2(K + 4) + 1)) \geq M(N, 2^{-(K+2)}) \geq R(N, 2K + 1)$ so that

$$\lim_{N \to \infty} \frac{\log(R(N, 2(K + 4) + 1))}{N} \leq \lim_{N \to \infty} \frac{\log(M(N, 2^{-(K+2)}))}{N} \leq \lim_{N \to \infty} \frac{\log(R(N, 2K + 1))}{N} .$$

For $K \to \infty$, the left and right limits converge to the same number. The limit in the middle is the topological entropy.