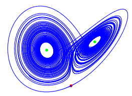
FRACTALS — Math118, O. Knill

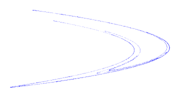
ABSTRACT. In order to define a strange attractor, we have to look at the notion of a "fractal", a set of fractional "dimension". The term fractal had been introduced by Benoit Mandelbrot in the late 70ies. We will see more about fractals later in this course, when we look at complex maps.

STRANGE ATTRACTOR. An **attracting set** of a differential equation $\dot{x} = F(x)$ or map $x \to T(x)$, is called a **strange attractor**, if it has **fractal dimension** (we will define that below), **sensitive dependence on initial conditions** (positive Lyapunov exponent) and which has an **indecomposable physical measure** which means that for almost all initial conditions x_0 and all continuous functions f, the limit $\frac{1}{t}\int_0^t f(T_s(x_0))\, ds$ rsp. $\frac{1}{n}\sum_{k=1}^n f(T^k(x)$ exists and depends only on f and not x_0.

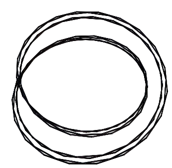
The Lorenz attractor: the dimension is numerically around 2.05 (Doering Gibbon 1995), 2.0627160 (Viswanath, 2002), The in-decomposability (technically called "SRB measure") (Tucker, 2002).



The Henon attractor: the dimension is measures 1.36 (Grebogi, Ott, Yorke, 1987) The in-decomposability had been shown (Benedicks and Carlson, 1991).



The Solenoid: This is a toy attractor, for which all the properties can be proven. It is a strange attractor for a map in space.

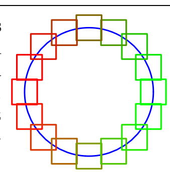


DIMENSION. Let X be a a set in Euclidean space. Define the s-volume of accuracy r of a set X as $h_{s,r}(X) = nr^s$, where n is the smallest number of cubes of side length r needed to cover X. The **s-volume** is the limit $h_s(X) = \lim_{r\to 0} h_{s,r}(X)$. The **box counting dimension** is defined as the limiting value s, where $h_s(X)$ jumps from 0 to infinity.

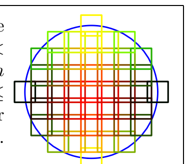
LINE SEGMENT. A **line segment** of length 1 in the plane can be covered with n intervals of length $1/n$ and $h_{s,r}(X) = n(1/n^s)$. For $s < 1$ this converges to 0, for $s > 1$, it converges to infinity. The dimension is 1.

SQUARE. A **square** X of a plane of area 1 in space can be covered with n^2 cubes of length $1/n$ and $h_{s,r}(X) = n^2(1/n^s)$ which converges to 0 for $s < 2$ and diverges for $s > 2$. The dimension is 2.

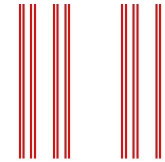
CIRCLE. A **circle** or radius 1 can be covered with with $2\pi n$ squares of length $1/n$ and $h_{s,r}(X) = 2\pi n(1/n^s)$. For $s < 1$ this converges to 0, for $s > 1$, it converges to infinity. The dimension is 1.



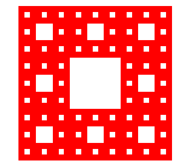
DISC. A **disc** of radius 1 in space can be covered with $\pi n^2/4 < N < \pi n^2$ squares of length $1/n$ and $\pi(n^2/4)/n^2 \leq h_{s,r}(X) \leq \pi n^2/n^s$ which converges to 0 for $s < 2$ and diverges for $s > 2$. The dimension is 2.



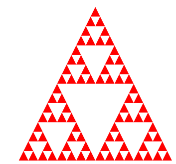
THE CANTOR SET. The **Cantor set** is constructed recursively by dividing the interval $[0,1]$ into 3 equal intervals and cutting away the middle one repeating this procedure with each of the remaining intervals etc. At the k'th stop, we need 2^k intervals of length $1/3^k$ to cover the set. The s-volume $h_{s,3^{-k}}(X)$ of accuracy $1/3^k$ is $2^k/3^{sk}$. It goes to zero if $s < 2/3$ and diverges for $s > \log(2)/\log(3)$.



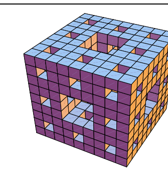
SHIRPINSKI CARPET. The **Shirpinski carpet** is constructed recursively by dividing a square in the plane into 9 equal squares and cutting away the middle one, repeating this procedure with each of the remaining squares etc. At the k'th step, we need 8^k squares of length $1/3^k$ to cover the carpet. The s-volume $h_{s,1/3^k}(X)$ of accuracy $1/3^k$ is $8^k(1/3^k)^s$ which goes to 0 for k approaching infinity if s is smaller than $d = \log(8)/\log(3)$ and diverges for s bigger than d. The dimension of the carpet is $d = \log(8)/\log(3) = 1.893$ a number between 1 and 2. It is a fractal.



SHIRPINSKI GASKET The **Shirpinski gasket** is constructed recursively by dividing a triangle in the plane into 4 equal triangles and cutting away the middle one, repeating this procedure with each of the remaining squares etc. At the k'th step, we need 3^k triangles of side length $1/2^k$ to cover the gasket. The s-volume $h_{s,1/2^k}(X)$ of accuracy $1/2^k$ is $8^k(1/2^k)^s$ which goes to 0 for k approaching infinity if s is smaller than $d = \log(3)/\log(2)$ and diverges for s bigger than d. The dimension of the gasket is $d = \log(3)/\log(2)$, a number between 1 and 2.



MENGER SPONGE. The three-dimensional analogue of the Cantor set in one dimensions and the Shirpinski carpet. One starts with a cube, divides it into 27 pieces, then cuts away the middle third along each axes. It is your task to compute the dimension. Note that the faces of the Menger sponge are decorated by Shirpinski Carpets.



THE PROBLEMS OF THE DEFINITION. If one takes the above definition, then the dimension of the set of rational numbers in the interval $[0,1]$ is equal to 1. A better definition, the **Hausdorff dimension** is needed. We include that definition below but it is a bit more complicated. The problem with the box counting dimension is that the size of the cubes should be allowed to vary. This refinement is similar to the change from the **Riemann integral** to the **Lebesgue integral**. c

HAUSDORFF MEASURE. Let (X,d) be a metric space. Denote by $|A| = \sup_{x,y\in A} d(x,y)$ the **diameter** of a subset A. Define for $\epsilon > 0, s > 0$

$$h^s_\epsilon(A) = \inf_{\mathcal{U}_\epsilon} \sum_{U\in\mathcal{U}_\epsilon} |U|^s ,$$

where $\mathcal{U}_\epsilon$ runs over all countable open covers of A with diameter $< \epsilon$. Such covers are also called **ϵ-covers**. The limit

$$h^s(A) = \lim_{\epsilon\to 0} h^s_\epsilon(A)$$

is called the $s-$ **dimensional Hausdorff measure** of the set A. Note that this limit exists in $[0,\infty]$ (it can be ∞), because $\epsilon \mapsto h^s_\epsilon(A)$ is increasing for $\epsilon \to 0$.

LEMMA: If $h^s(A) < \infty$, then $h^t(A) = 0$ for all $t > s$. Take $\epsilon > 0$ and assume $\{U_j\}_{j\in\mathbf{N}}$ is an open ϵ-cover of A. Then

$$h^t_\epsilon(A) \leq \sum_j |U_j|^t \leq \epsilon^{t-s} \cdot \sum_j |U_j|^s .$$

Taking the infimum over all coverings gives

$$h^t_\epsilon(A) \leq \epsilon^{t-s} \cdot h^s_\epsilon(A) .$$

In the limit $\epsilon \to 0$, we obtain from $h^s(A) < \infty$ that $h^t(A) = 0$.

HAUSDORFF DIMENSION.
Either there exists a number $\dim_H(A) \geq 0$ such that

$$s < \dim_H(A) \quad \Rightarrow \quad h^s(A) = \infty \ ,$$
$$s > \dim_H(A) \quad \Rightarrow \quad h^s(A) = 0$$

or for all $s \geq 0$, $h^s(A) = 0$. In the later case, one defines $\dim_H(A) = \infty$.
The number $\dim_H(A) \in [0, \infty]$ is called the **Hausdorff dimension** of A.

FRACTAL. A **fractal** is a subset of a metric space which has finite non-integer Hausdorff dimension.

The Hausdorff dimension is in general difficult to calculate numerically. The central difficulty is to determine the infimum over $\sum_i |U_i|^t$, where $\mathcal{U} = \{U_i\}$ is an ϵ-cover of A. The box-counting dimension simplifies this problem by replacing arbitrary covers by sphere covers and so to replace the terms $|U_i|^t$ by ϵ^t. The prize one has to pay is that one can no more measure all bounded sets like this. In general, the upper and lower limits differ.

UPPER AND LOWER CAPACITY. Given a compact set $A \subset X$. Define for $\epsilon > 0$, $N_\epsilon(A)$ as the smallest number of sets of diameter ϵ which cover A. By compactness, this is finite. Define the **upper capacity**

$$\overline{\dim}_B(A) = \limsup_{\epsilon \to 0} \frac{\log(N_\epsilon(A))}{-\log(\epsilon)}$$

and analogous the **lower capacity** $\underline{\dim}_B(A)$, where lim sup is replaced with lim inf. If the lower and upper capacities coincide, the value $\dim_B(A)$ is called **box counting dimension** of A.

CAPACITY DIMENSION. If the lower and upper capacity are the same, one calls it the **capacity dimension**.

BOX COUNTING DIMENSION. Cover $\mathbf{R}^n$ by closed square boxes of side length 2^{-k}. and let $M_k(A)$ be the number of such boxes which intersect A. Define the box counting dimension

$$\dim_B(A) = \lim_{k \to \infty} \frac{\log(M_k(A))}{\log(2^k)} \ .$$

If the capacity dimension exists, then it is equal to the box counting dimension.

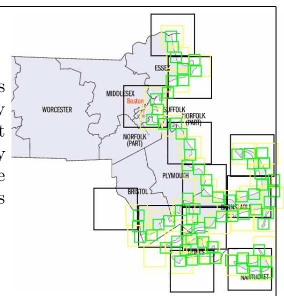
PROOF: Any set of diameter 2^{-k} can intersect at most 2^n grid boxes. On the other hand, any box of side 2^{-k} has diameter smaller than 2^{-k+1}. There exists therefore a constant C such that

$$C^{-1} \cdot M_k(A) \leq N_{2^{-k}}(A) \leq C \cdot M_k(A) \ .$$

Therefore

$$\lim_{k \to \infty} \frac{\log(M_k(A))}{\log(2^k)} = \lim_{k \to \infty} \frac{\log(N_{2^{-k}}(A))}{\log(2^k)} \ .$$

SELF-SIMILARITY. The computation of the dimension in the example objects was easy because they are **self-similar**. A part of the object is when suitably scaled equivalent to the object. We we will see more about this when we look at iterated function systems. To measure or estimate the dimension of an arbitrary object, one has to count squares. As an illustration of fractals in nature, one often takes coast lines. A rough estimate of the coast of Massachusetts leads to a dimension 1.3.



HISTORY.

The Cantor set is named after George Cantor (1845-1918), who was putting down the foundations of set theory. Ian Stewart writes in "Does God Play Dice", 1989 p. 121:
"The appropriate object is known as the Cantor set, because it was discovered by Henry Smith in 1875. The founder of set theory, George Cantor, used Smith's invention in 1883. Let's fact it, 'Smith set' isn't very impressive, is it?

The Hausdorff dimension has been introduced in 1919 by **Felix Hausdorff** (1868-1942).

Abram Besicovitch, around 1930, worked out an extensive theory for sets with finite Hausdorff measure.

The name "fractal" had been introduced only much later by Benoit Mandelbrot (1924-) in 1975.

The Sierpinski carpet was studied by **Waclaw Sierpinski** in 1916. He proved that it is universal for all one dimensional compact objects in the plane. This means that if you draw a curve in the plane which is contained in some finite box, however complicated it might be and with how many self-intersections you want, there is always a part of the Shirpinski carpet which is topologically equivalent to this curve.

This might not look so surprising but this result is not true for the Shirpinski gasket. The Menger Sponge was studied by **Klaus Menger** in 1926. He showed that it is universal for all one dimensional objects in space. This means whatever complicated curve you draw in space, you find a part of the Menger sponge, which is topologically equivalent to it.