FIXED POINT THEOREMS

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ABSTRACT. Fixed point theorems are important in dynamics.

BANACHS FIXED POINT THEOREM.

A contraction T in a complete metric space X has a fixed point.

This theorem can be used for example to prove the existence of solutions to differential equations.

BROWERS FIXED POINT THEOREM.

Every continuous map T from the unit ball $D^n = \{x \in R^n \mid ||x|| \leq 1\}$ onto itself has a fixed point.

SKETCH OF PROOF FOR $n = 2$. If $T(x) \neq x$ for all $x \in D^n$, one can find a continuous map g from D^n to its boundary S^{n-1}: the point $g(x)$ is the intersection of the line through x and $T(x)$ with S^{n-1}. This map is the identity on the boundary. If such a map existed, one could smooth it. We would have a smooth map from the interior of D to S^{n-1}. For most $y \in S^{n-1}$ the set $S^{-1}(y)$ is a curve in D which begins and ends at y. The region it contains must by continuity also be mapped to y and $S^{-1}(y)$ would contain a disc and can not be a curve.

REMARK TO 1D: The Brower fixed point theorem in one dimensions, (D^1 is an interval $[a, b]$) follows from the intermediate value theorem: Since $T(a) \geq a, T(b) \leq b$, the function $g(x) = T(x) - x$ satisfies $g(a) > 0$ and $g(b) < 0$. It must have a root. This root is the fixed point. theorem.

KAKUTANI FIXED POINT THEOREM.

A continuous map T on a compact convex set D in normed space X has a fixed point.

(We formulated a special case only. One can weaken the assumption on X as well as relax the condition that T must be a map: it can also be a correspondence for which $T(x)$ is a convex subset of X.) While von Neumann used Browers fixed point theorem, John Nash was among the first to use Kakutani's Fixed Point Theorem in game theory, where fixed point are often **equilibria**.

POINCARE BIRKHOFF THEOREM.

An area-preserving transformation on the annulus, which moves boundary circles in the opposite directions has at least two distinct fixed points.

Poincare had conjectured this but could no more prove it. The conjecture was therefore called **Poincarés last theorem**. It was George Birkhoff who proved it in 1917.

APPLICATION TO BILLIARDS.

COROLLARY. For a billiard in a smooth convex table, there are at least 2 periodic orbits of type $0 < p/q < 1$ meaning that T^q winds around the table p times.

PROOF. The map T^q leaves one boundary of the annulus $X = T^1 \times [-1, 1]$ fixed, the other boundary is turned around q times. Now define $S(x, y) = (x - 1, y)$ which rotates every point once around. Now, $T^q S^{-p}$ rotates one side of the boundary by $-2\pi p$ and the other side of the boundary by $2\pi(q - p)$. Since the boundary is now turned into different directions, there are fixed points of $T^q S^{-p}$. For such a fixed point $T^q(x, y) = S^p(x, y)$ which is what we call orbit of type $0 < p/q < 1$.

APPLICATIONS TO DUAL BILLIARDS.

COROLLARY. For exterior billiard at a smooth convex table, there are at least 2 periodic orbits of type $0 < p/q < 1/2$ meaning that T^q winds around the table p times.

Periodic orbits with small rotation numbers p/q are close to the table, periodic orbits with rotation number close to $1/2$ are far away from the table.