2/7/2005: THE LOGISTIC MAP — Math118, O. Knill

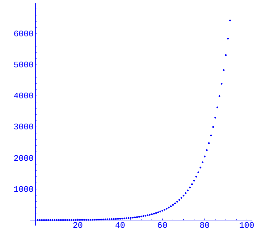
ABSTRACT. Our first dynamical system is the **logistic map** $f(x) = cx(1-x)$, where $0 \leq c \leq 4$ is a **parameter**. It is an example of an **interval map** because it can be restricted to the interval $[0,1]$.

You can read about this dynamical system on pages 14-16, pages 57-60, pages 198-199 as well as from page 299 on in the book. On this lecture, we have a first look at interval maps. We will focus on the logistic map, study periodic orbits, their stability as well as stability changes which are called bifurcations.

A FIRST POPULATION MODEL. In a simplest possible population model, one assumes that the population growth is proportional to the population itself. If x_n is the population size at time n, then $x_{n+1} = T(x_n) = x_n + ax_n = cx_n$ with some constant $a > 0$. We can immediately give a closed formula for the population x_n at time n:

$$x_n = T^n(x) = c^n x_0 \, .$$

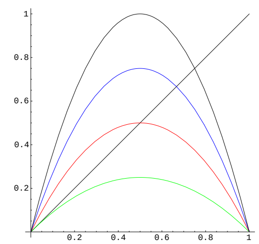
We see that for $c > 1$, the population grows **exponentially** for $c < 1$, the population shrinks exponentially.



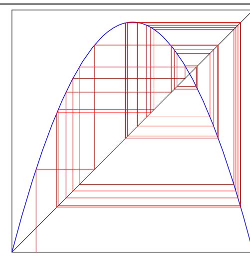
DERIVATION OF THE LOGISTIC POPULATION MODEL. If the population gets large, food becomes sparse (or the members become too shy to reproduce ...). In any case, the growth rate decreases. This can be modeled with $y_{n+1} = cy_n - dy_n^2$. Using the new variable $x_n = (c/d)y_n$, this recursion becomes

$$x_{n+1} = cx_n(1-x_n) \, .$$

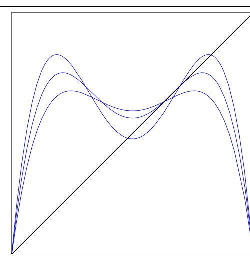
To the right, we see a few graphs of $f_c(x) = cx(1-x)$ for different c's. The intersection of the graph with the diagonal reveals **fixed points** of f_c. You see that 0 is always a fixed point. The graph has there the slope $f'(0) = c$. For $c > 1$, there exists a second fixed point $x = 1 - \frac{1}{c}$.



INTERVAL MAPS. If $f : [0,1] \to [0,1]$ is a **map** like $T(x) = 3x(1-x)$ and $x \in [0,1]$ is a point, one can look at the successive **iterates** $x_0 = x, x_1 = T(x), x_2 = f^2(x) = T(T(x)), \ldots$. The sequence x_n is called an **orbit**. If $x_n = x_0$, then x is called a **periodic orbit** of period n or n-cycle. If there exists no smaller $n > 0$ with $x_n = x$, the integer n is called the **true period**. A **fixed point** of f is a point x such that $f(x) = x$. Fixed points of f^n are periodic points of period n. The fixed points of f are obtained by intersecting the graph $y = f^n(x)$ with the graph $y = x$. The iterates of an **interval map** can visualized with a **cobweb construction**: connect (x,x) with $(x, f(x))$, then go back to the diagonal $(f(x), f(x))$ and iterate the procedure.

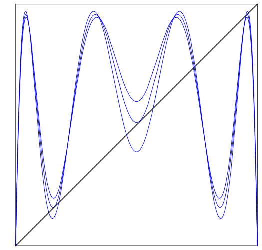


STABILITY OF PERIODIC POINTS. If x_0 is a fixed point of a differentiable interval map f and $|f'(x_0)| > 1$, then x_0 is **unstable** in the following sense: a point close to x_0 will move away from x_0 at first because linear approximation $T(x) \sim x_0 + f'(x_0)(x - x_0)$ shows that $|f(x) - x_0| \sim |f'(x_0)||x - x_0| > |x - x_0|$ near x_0. On the other hand, if $|f'(x)| = 1$, then x_0 is stable. For periodic points of period n, the stability is defined as the stability of the fixed point of f^n. The picture to the right shows situations, where $f'(x_0) < 1, f'(x_0) = 1, f'(x_0 > 1$ at a fixed point. The parameter at which the stability changes will be denoted a bifurcation.

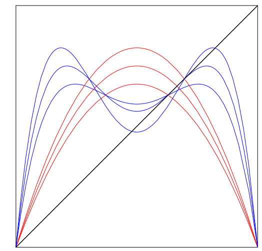


REMEMBER THE IMPORTANT FACT: if $f(x) = x$ is a fixed point of f and $|f'(x)| < 1$, then the fixed point is **stable**. It attracts an entire neighborhood. If $|f'(x)| > 1$, then the fixed point is **unstable**. It repells points in a neighborhood.

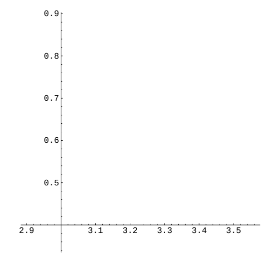
BIFURCATIONS. Let f_c be a **family of interval maps**. Assume that x_c is a fixed point of f_c. If $|f_c'(x_c)| = 1$, then c_0 is called a **bifurcation parameter**. At such a parameter, the point x_c can change from stable to unstable or from unstable to stable if c changes. At such parameters, it is also possible that new fixed points can appear. Different type of bifurcations are known: **saddle-node bifurcation** (also called **blue-sky** or **tangent** bifurcation). They can be seen in the picture to the right). **Flip bifurcations** (also related to **pitch-fork bifurcation**) lead to the **period doubling bifurcation** event seen below.



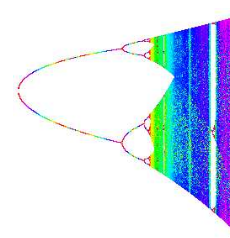
PERIOD DOUBLING BIFURCATION. Period doubling bifurcations happen for parameters c for which $(f_c^n)'(x_c) = -1$. The graph of f_c intersects the diagonal in one point, but the graph of f_c^2 which has slope 1 at x_0 starts to have three intersections. Two of the intersections belong to newly formed periodic points, which have twice the period. To the right, we see a simultaneous view of the graphs of f_c and f_c^2 for $c = 2.7, c = 3.0, c = 3.3$. You see that f_c keeps having one fixed point throughout the bifurcation. But f_c^2, which has initially one fixed point starts to have 3 fixed points! The middle one has minimal period 1, the other two are periodic points with minimal period 2.



BIFURCATION DIAGRAM. The **logistic map** $f_c(x) = cx(1-x)$ always has the fixed point 0. For $c > 1$, there is an additional fixed point $x = 1 - 1/c$. Because $f'(0) = c$, the origin is stable for $c < 1$ and unstable for $c > 1$. At the other fixed point, $f'(x) = c - 2cx = c - 2c(1 - 1/c) = 2 - c$. It is stable for $1 < c < 3$ and unstable for $c > 3$. The point $c = 3$ is a bifurcation. It is called a **flip bifurcation**. Because a periodic point of period 2 is created, it is called a **period doubling bifurcation**. To see what happens with the periodic point of period 2, we look at $f^2(x) - x = c^2x(1-x)(1 - cx(1-x)) - x$ which has the roots $(c + 1 \pm \sqrt{(c-3)(c+1)})/(2c)$ which are real for $c > 3$. Its stability can be determined with $(f^2)'(x) = f'(x)f'(f(x)) = 4 + 2c - c^2$. This shows that the 2-cycle is stable for $3 < c < 1 + \sqrt{6}$. At the parameter $1 + \sqrt{6}$ it bifurcates and gives rise to a periodic orbit of period 4.



FEIGENBAUM UNIVERSALITY. We have computed the first bifurcation points $c_1 = 3, c_2 = 1 + \sqrt{6}$. The successive period doubling bifurcation parameters c_k have the property that $\frac{c_{k+1} - c_k}{c_{k+2} - c_{k+1}}$ converges to a number $\delta = 4.69920166$. It was **Mitchell Feigenbaum**, who realized that this number is **universal** and conjectured how it could happen using a **renormalization picture**. In 1982, **Oscar Lanford** proved these Feigenbaum conjectures: for a class of smooth interval maps with a single quadratic maximum, the limit δ exists and is universal: that number does not depend on the chosen family of maps. It works for example also for the family $f_c(x) = c\sin(\pi x)$. The proof demonstrates that there is a fixed point g of a **renormalization map** $\mathcal{R}f(x) = \alpha f^2(\alpha x)$ in a class of interval maps. The object which is mapped by $\mathcal{R}$ is a map!



HISTORY. Babylonians considered already the rotation$f(x) = x + \alpha$ mod 1 on the circle. Since the 18th century, one knows the Newton-Rapson method for solving equations. Already in the 19th century **Poincaré** studied circle maps. Since the beginning of the 20th century, there exists a systematic theory about the iteration of maps in the complex plane (Julia and Fatou), a theory which applies also to maps in the real. In population dynamics and finance growth models $x_{n+1} = f(x_n)$ appereard since a long time. It had been popularized by theoretical biologists like Robert May in 1976. Periodic orbits of the logistic map were studied for example by N. Metropolis, M.L. Stein and P.R. Stein in 1973. Universaility was discovered numerically by Feigenbaum (1979) and Coullet-Tresser (1978) and proven by Lanford in 1982.