STRANGE SINGULARITIES AND ORBITS **Math118, O. Knill**

ABSTRACT. Non-collision singularities are possible in the Newtonian n-body problem by careful construction. Also the construction of special solutions to the n-body problems is an art.

PAINLEVES CONJECTURE: Painlevé asked in his Stockholm lectures of 1895: for $n > 3$, do there exist solutions of the Newtonian n-body problem with singularities that are not due to collisions?

HISTORY. Zeipels theorem showed that singularities of the Newtonian n-body problem are either collisions or configurations for which particles escape to infinity in finite time. Poincaré seems have considered this question already, even so he never wrote it down. Painlevé gave Poincarée credit for having asked that some $x_i(t)$ might go to infinity or oscillate wildly like $\sin(1/(t-\tau)$ as t converges to the singularity. Painlevé himself proved that non-collision singularities do not exist for the three body problem. Painlevés question whether non-collision singularities can occur, stayed open until Jeff Xia constructed non-collision singularities in 1992. (By the way, Xia was at Harvard from 1988-1990, so some of the final polishing of this paper could have been done here). Another mathematician, Joseph Gerver, had also been in the race but considered a planar approach, where the number of particles is large. John Mather and Richard Mc Gehee had already in 1974 shown that particles can escape to infinity but their construction on the one dimensional line and binary collisions were allowed. While it is known that for four bodies, non-collision singularities have measure zero, one does not know whether they exist. There is a construction of a planar 4 body situation of Gerver from 2003 which suggests that the answer could be yes.

A THEOREM OF PAINLEVE.

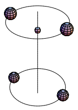
THEOREM (1897) There are no non-collision singularities in the three body problem.

PROOF. The Lagrange-Jacobi equation is $\ddot{I} = U + 2H$, where $I = \sum_{j=1}^{3} m_j r_j^2$ is the **moment of inertia**. I is a measure of diameter of the triangle defined by the positions of the three particles. These equations imply that whenever two particles come close, the triangle they span has to become large. By the triangle inequality, two sides of the triangle are then large. The Sundman-van Zeipel lemma assured that $I(t) \to I^*$ for $t \to \tau$ with $I^* = \infty$ if there is a non-collision singularity. Assuming $I(t) \to \infty$ for $t \to \tau$ we have $I(\ddot{t}_k) \to \infty$ for some sequence of times $t_k \to \tau$ which means $U(t_k) \to \infty$. This implies that two of the three particles must come close to each other. In the same time, the third "lonely" particle has to be far away from these two particles because $I(t) \to \infty$. Because the acceleration of the lonely particle and the center of mass of the binary both stay bounded for $t \to \tau$, these positions converge to a definite finite value for $t \to \tau$. The collision assumption means that the binary system collides for $t = \tau$ but at a finite distance from the third particle. Consequently $I(\tau) = I^* < \infty$, which is in direct contradiction to the assumption $I^* = \infty$.

THEOREM OF XIA.

Non-collision singularities exist in the Newtonian 5 body problem. There are initial conditions for the Newtonian 5 body problem in which the bodies escape to infinity in finite time.

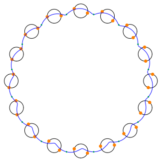
BASIC IDEA. The setup is to add a second binary solar system to the Sitnikov system. The planet moving on the z-axes visits alternatively the two binary systems. The timing is done in such a way that the planet will bounce back accelerated after visiting one of the systems. The energy is drawn from the potential energy of the two binary systems which move closer and closer together. The four suns have all the same mass. The upper and lower "solar systems" have opposite angular momentum and their "Kepler orbits" are highly eccentric.



THEOREM OF GERVER. Joseph Gerver proved a theorem for the planar case:

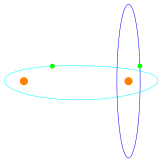
THEOREM. For large n, non-collision singularities exist for the planar n-body problem

BASIC IDEA. There are 3N bodies in the plane. The configurations are symmetric with respect to rotations by $2\pi/N$. There are N binary systems in which all suns have the same mass. There are N planets which move from one pair to the other. The successive time spans, which the planets need to jump from one to the next system forms a sequence Δ_k with the property that $\sum_k \Delta_k < \infty$.



GERVERS SUGGESTION: Are there planar four body configurations in which particles escape to infinity in finite time?

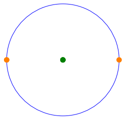
Gervers model contains two planetary systems: there are two suns S_1, S_2 with large mass and two planets P_1, P_2 with small mass. Planet P_2 circles sun S_2 in an elliptical orbit. Planet P_1 circles around Sun S_1 and visits the second planetary system, where it alternatively gains angular momentum and energy.



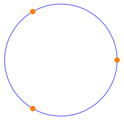
SPECIAL SOLUTIONS. An interesting research topic is the search for special solutions of the 3 body problem.

EQUILIBRIUM SOLUTIONS. Whenever we studied differential equations, we were interested in **equilibrium solutions**, stationary solutions. Are there equilibrium solutions for the Newtonian n body problem? The answer is no: from $\ddot{x}_k = 0$ would imply that $U_{x_k} = 0$ and from Eulers theorem on homogeneous functions that $-U = \sum_{j=1}^{n} x_k U_{x_k} = 0$. But the potential U is clearly positive everywhere.

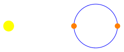
EULERS SOLUTIONS (1767) Euler was the first who found special solutions to the three body problem. In these solutions, the three bodies rotate on circles but remain on a line. The Euler solution and the Lagrange solution below are the only solutions for which the particles move uniformly along circular orbits in a fixed plane.



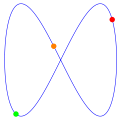
LAGRANGIAN SOLUTIONS (1772). The three bodies are on an equilateral triangle. This system appears in nature: the Trojan asteroids together with Jupiter and the sun essentially move according to this. Lagrange, who found this solution did not think this has any significance in astronomy.



HILLS SOLUTIONS. These are configurations resembling the Earth-Moon-Sun system. Two bodies move closely around each other while both of them circle a third body.



MOORE CHOREOGRAPHIES. Three bodies of equal mass follow each other on a figure eight type orbit. These solutions have been discovered by Cristopher Moore in 1993 through computer calculations.



LITERATURE. A vivid account on the history of non-collision singularities also containing many annectotes about the discovery is the book "Celestial Encounters" by Florin Diaco and Philp Holmes. The article "Off to infinity in Finite Time" by Donald Saari and Jeff Xia gives a nice summary. For a suggestion, how a four body noncollision singularity might work, see Joseph Gervers article "Non collision Singularities: Do four bodies Suffice?".