2/11/2005: ENTROPY AND CHAOS — Math118, O. Knill

ABSTRACT. We look today at some notions of "chaos". One definition is the positivity of a number called the **positive entropy**, an other is the positivity of the Lyapunov exponent for every orbit which is not eventually periodic. An other definition is "chaos in the sense of Devaney". The Ulam map or the tent maps are examples for which we know that this type of chaos happens.

A DEFINITION OF CHAOS.
A purely topological notion of chaos which applies also to map with no differentiability is the **definition of Devaney**:

> A map $T : [0,1] \to [0,1]$ is called **chaotic**, if there is a dense set of periodic orbits and if there exists an orbit which is dense.

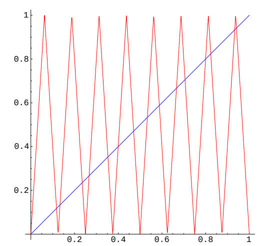
A set Y is **dense** in $[0,1]$ if there is no interval which has empty intersection with Y.
EXAMPLES. a) the set of rational numbers is dense in $[0,1]$.
b) the set of irrational numbers is dense in $[0,1]$.
c) The set of numbers $\{1/n \mid n = 1,2,...\}$ is not dense in the interval.
d) Consider **Champernown's number** $x = 0.12345678910111213141516171819202122324...$ (do you see the pattern?), and the map $T(x) = 10x \text{ mod} 1$. Then $T(x) = 0.2345678910111213141516171819202122324...$ and $T^2(x) = 0.345678910111213141516171819202122324...$ etc. Can you see why the orbit of x under the map T is dense in the interval $[0,1]$?

THE ULAM MAP IS CHAOTIC. We only state this theorem now. We will put later in this course put together some tools to prove it.

> THEOREM. The map $f_4(x) = 4x(1-x)$ is chaotic in the sense of Devaney.

To have Devaney chaos, one needs to have an initial point, which visits each interval as well as to find for each interval a periodic orbit which visits that interval.
Because the Ulam map is conjugated to the tent map, we need only to prove the claim for the tent map.
In the homework, you see the density of the periodic points by understanding the graphs of the iterates of the map.

The problem to construct a dense periodic point will be solved later.



TOPOLOGICAL ENTROPY. Let $P_n(f)$ be the number of periodic points of true period n. Define the topological entropy of the map as

$$p(f) = \limsup_{n\to\infty} \frac{1}{n} \log |P_n(f)| ,$$

where the limits $p(f) = +\infty$ and $p(f) = -\infty$ are also allowed.
The topological entropy measures the growth of the number of periodic points. Similarly as the Lyapunov exponent, it measures how "complex" the map is.

EXAMPLES. The map defined by $\tilde{f}(x) = 2x \bmod 1$ has the topological entropy $p(f) = \log(2)$ because $P_n(f) = 2^n$.

DYNAMICAL ZETA FUNCTION Related to topological entropy is the **dynamical ζ function** which is defined as

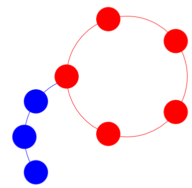
$$\log(\zeta_f(z)) = \sum_{n=1}^{\infty} \frac{P_n(f)}{n} z^n ,$$

where z is a real (or complex) variable. The series converges if $P_n(f)$ is finite for all n and for all complex numbers $|z| < e^{-p(f)}$. If $p(f) = -\infty$ and $P_n(f)$ is finite for all n, then $\zeta_f(z)$ is defined for all z.

EXAMPLE. The dynamical zeta function satisfies $\log(\zeta_f(z)) = \sum_{n=1}^{\infty} \frac{2^n}{n} z^n$. Because $\sum_{n=1}^{\infty} x^n = 1/(1-x) - 1$ and integration gives $\sum_{n=2}^{\infty} x^n/n = -\log(1-x) - x$ we have $\sum_{n=1}^{\infty} x^n/n = -\log(1-x)$. We see that $\log(\zeta_f(z)) = -\log(1-2z)$ and

$$\zeta_f(z) = 1/(1-2z) .$$

EVENTUALLY PERIODIC ORBITS. If an orbit has the structure $x_0, x_1, x_2, ..., x_m, x_{m+1}, x_{m+2}, ..., x_{m+n} = x_m$, it is called **eventually periodic**. Eventually periodic orbits appear often in dynamical systems which are not invertible.



EXAMPLES:
1) The point $x_0 = 1$ of the logistic map $f_c(x) = cx(1-x)$ is eventually periodic. It is actually eventually fixed. We have

$$x_0 = 1, x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$$

2) The point $x_0 = 7/10$ is eventually periodic for $T(x) = 1 - 2|x - 1/2|$.

EVENTUALLY PERIODIC POINTS FOR THE TENT MAP.

> THEOREM. x is eventually periodic if and only if $x = p/q$ is rational.

PROOF.
Since $T(x) = 2x$ or $T(x) = 2 - 2x$ we have

$$T(x) = integer + 2x$$

$$T^2(x) = integer + 2^2 x$$

$$T^n(x) = integer + 2^n x$$

If $T^n(x) = T^m(x)$ then $k + 2^n x = l + 2^m x$ so that $x = (k-l)/(2^n - 2^m)$ and x is a rational number.

(ii) To see the other direction, lets assume now that $x = p/q$ is rational. Then, $T(x) = 2p/q$ or $T(x) = 2 - 2p/q = 2(q-p)/q$. In any case, $T(x)$ is again of the form k/q. Repeating this argument shows that $T^n(x)$ is of the form k/q. There are only finitely many fractions of the form k/q and x therefore has to be eventually periodic.

REMARK. It needs a bit of combinatorial thought to figure out, when an orbit is eventually periodic and when it is actually periodic. Here is the answer (without a proof):

> THEOREM. $x = p/q$ is periodic for the tent map if and only p is an even integer and q is an odd integer.

EXAMPLES:
1) $x = 4/5$ is a periodic point of period 2.
$x_0 = 4/5, x_1 = 2/5, x_2 = 4/5$ etc

2) $x = 5/7$ is an eventually periodic point.
$x_0 = 5/7, x_1 = 4/7, x_2 = 6/7, x_3 = 2/7, x_4 = 4/7$.

EVENTUALLY PERIODIC POINTS FOR THE ULAM MAP. The conjugation between the two maps T and S matches periodic points of T to periodic points of S and "eventually periodic points" of T with eventually periodic points of S.

EXAMPLE: Because $x_0 = 5/7$ is an eventually periodic point for the tent map, the point $y_0 = U^{-1}(5/7) = (1 - \cos(5\pi/7))/2$ is the initial condition for an eventually periodic point for the Ulam map.

$x_0 = 5/7$	$x_1 = 4/7$	$x_2 = 6/7$	$x_3 = 2/7$	$x_4 = 4/7$
$\uparrow$ U	$\uparrow$ U	$\uparrow$ U	$\uparrow$ U	$\uparrow$ U
$y_0 = 0.811745$	$y_1 = 0.61126$	$y_2 = 0.950$	$y_3 = 0.1882$	$y_4 = 0.61126$