2/2/04 WHAT ARE DYNAMICAL SYSTEMS? Math118, O.Knill

ABSTRACT. We discuss in this lecture, what dynamical systems are and where the subject is located within mathematics.

A FIRST DEFINITION.
The theory of dynamical systems deals with the **evolution of systems**. It describes **processes in motion**, tries to **predict the future** of these systems or processes and understand the **limitations of these predictions**.

RELEVANCE OF DYNAMICAL SYSTEMS.
To see that dynamical systems are relevant, one has just to look at a few news stories which broke during the last few weeks:

- Tsunami damage prediction
- Metor path computation
- Currents in the sea
- Landing of the Cassini probe on Titan
- Roulette ball prediction
- Statistics of digits in π
- Global earth temperature prediction

A FANCY DEFINITION.
Mathematically, any semigroup G acting on a set is a dynamical system. A **semigroup** $(G, \star)$ is a set G on which we can add two elements together and where the **associativity law** $(x \star y) \star z = x \star (y \star z)$ holds. The action is defined by a collection of maps T_t on X. It is assumed that $T_{t \star s} = T_t \circ T_s$, where $\star$ is the operation on G (usually addition) and $\circ$ is the composition of maps.

CLASSES OF DYNAMICAL SYSTEMS:

Time G (semigroup)	**Action**
Natural numbers $(N, +)$	Maps
Integers $(Z, +)$	Invertible maps
Positive real numbers $(R^+, +)$	Semiflows (some PDE's)
Real numbers $(R, +)$	Flows (Differential equations)
Any group $(G, \star)$	Representations
Lattice $(Z^n, +)$	Lattice gases, Spin systems
Euclidean space $(R^n, +)$	Tiling dynamical systems
Free group $(F_n, \circ)$	Iterated function systems

TWO IMPORTANT CASES OF ONE DIMENSIONAL TIME. We mention the general definition to stress that the ideas developed for one dimensional time generalize to other situations. Because physical time is one dimensional, the important cases for us are definitely **discrete and continuous dynamical systems**:

dynamics of **maps** defined by transformations	dynamics of **flows** defined by differential equations

DYNAMICAL SYSTEMS AND THE REST OF MATH. All areas of mathematics are linked together in some way or an other. Intersections of fields like algebraic topology, geometric measure theory, geometry of numbers or algebraic number theory can be considered full blown independent subjects. The theory of dynamical systems has relations with all other main fields and intersections typically form subfields of both.

Algebra	Measure theory	Analysis
Topology	Probability theory	Geometry
Logic	Dynamics	Number Theory

EXAMPLES OF INTERSECTIONS OF DYNAMICS WITH OTHER FIELDS:

- Link with **algebra**: group theorists often look at the action of the group on itself. The action of the group on vector spaces defines a field called **representation theory**.
- Link with **measure theory**: in **ergodic theory** one studies a map T on a measure space (X, μ). Measure theory is one foundation of ergodic theory.
- Link with **analysis**: the study of **partial differential equations** or **functional analysis** as well as **complex analysis** or **potential theory**.
- Link with **topology**: the **Poincare conjecture** states that every compact three dimensional simply connected manifold is a sphere. The problem is currently attacked using a dynamical system on the space of all surfaces which is called the **Ricci flow**.
- Link with **geometry**: **Kleins Erlanger program** attempted to classify geometries by its symmetry groups. For example, the group of projective transformations on a projective space. A concrete dynamical system in geometry is the geodesic flow. An other connection is the relations of partial differential equations with intrinsic geometric properties of the space.
- Link with **probability theory**: sequences of **independent random variables** can be obtained using dynamical systems. For example, with $T(x) = 2x$ mod 1 and with the function f which is equal to 1 on $[0, 1/2]$ and equal to 0 on $[1/2, 1]$, $f(T^n(x))$ are independent random variables for most x.
- Link with **logic**: **logical deductions** in a proof or doing computations can be modeled as dynamical systems. Because every **computation** by a **Turing machine** can be realized as a dynamical system, there are fundamental limitations, what a dynamical system can compute and what not.
- Link with **number theory**: some problems in the theory of **Diophantine approximations** can be seen as problems in dynamics. For example, if you take a curve in the plane and look at the sequence of distances to nearest lattice points, this defines a dynamical system.
- A final link: a **category** X of mathematical objects has a semigroup G of **homomorphisms** acting on it (topological spaces have continuous maps, sets have arbitrary maps, groups, rings fields or algebras have homomorphisms, measure spaces have measurable maps). We can view each of these categories as a dynamical system. One can even include the category of dynamical systems with suitable homomorphisms. But this viewpoint is not a very useful in itself.