CONCLUSIONS	**Math118, O. Knill**

ABSTRACT. We summarize the main points of this course and add some didactical comments.

DYNAMICAL SYSTEMS. While the notion of dynamical systems can be defined in much greater generally, all dynamical systems considered here were either given by a map T on space X or by a differential equation $\dot{x} = f(x)$.

MATHEMATICAL STRUCTURES. The space X can carry different structures. It can be **topological**, **measure theoretical**, **combinatorial**, **geometrical** or **analytical**. Stressing the topological structure leads to topological dynamics, using an invariant measure reaches out to **probability theory** or **ergodic theory**, the geometrical structure is involved when dealing with differentiable functions and subject to **differential geometry**. Combinatorial structures come into play, when doing symbolic dynamics, when dealing with complexity or counting issues. The analytic structure is involved when the map can be extended to the complex, crossing the boundary to **complex analysis**, **algebraic geometry** or **potential theory**.

Topic	Examples	Key points
dynamical systems	semigroup action	the subject has relations with virtually any field of mathematics
1D dynamics	quadratic map, Ulam map	periodic points and their bifurcations, conjugation, Lyapunov exponents
2D dynamics	Henon map, Standard map	horse shoe construction, stability of periodic points, stable and unstable manifolds, Jacobean
2D differential equations	van der Pool equation, linear systems	Poincare-Bendixon
3D differential equations	Lorentz system	Poincare return map, Hopf bifurcation, Lyapunov function, fractals
billiards	polygons, ellipse, stadium	variational principle to construct orbits, effect for chaos
cellular automata	elementary 1D automata, life, lattice gases	topology of sequence space, attractor special solutions
complex dynamics	quadratic maps	Newton method, stability of periodic points, conjugation to normal form
symbolic dynamics	Baker map, full shift, Fibonacci shift, even shift	graphs from forbidden words, symbolic dynamics in general system
dynamics in number theory	irrational rotation, maps on finite sets	continued fraction expansion, dynamic logarithm problem, dynamical systems from curves
celestial mechanics	Kepler, Sitnikov, restricted planar 3-body problem	integrals, horse shoe construction, rotating coordinate systems
geodesic flow	plane, sphere, surfaces of revolution	surface billiards, integrals, caustics, calculus of variations

Some of main points I wanted to make in this course:

- Even deterministic systems lead to unpredictable or uncomputable situations.
- Some systems allow explicit solutions, other systems remain mysterious.
- The history of dynamical systems often sits at the heart of the history of mathematics or science.
- The subject has connections with many other fields of mathematics.
- Dynamical systems theory has many applications.
- There are many open problems left in the area of dynamical systems.

DIDACTICS. We have covered a lot of different topics. One could teach this course with the material from the first or second week, but in more depth. That would make sense too. I personally think that in a time where knowledge is accumulated at a tremendous speed, it makes sense to be trained also in the process of acquiring a lot of knowledge in a short time. Equally important is the ability to solve not so straightforward problems and to find creative solutions.

WHAT DID WE LEAVE OUT? Each of the topics could be extended to a full course. Important fields, which have not been touched at all: partial differential equations and systems in fluid dynamics in particular, systems with higher dimensional time as they appear in statistical mechanics, dynamical systems of algebraic origin. A large area for dynamics is also **game theory** or the theory of **neural networks**. Then there are problems of **statistical flavor** which deals with the problem to find the laws of the dynamical system from data. A particular case in statistics is to recover the space X the transformation T as well as the measure μ which produces the data. Finally, there are quantum versions of many dynamical systems considered so far. For billiards or surface billiards, the quantum problem is the study of the Laplacian on the surface with Dirichlet boundary conditions. The eigenvectors of the Laplacian v_n in the limit $n \to \infty$ have connections with the billiard or geodesic flow on the surface. **Quantum dynamical systems** can be obtained reformulating things first on a **function space**. For a topological dynamical system (X, T), consider the space $\tilde{X} = C(X)$ of all continuous functions on X. The map T induces a linear map $\tilde{T}$ on $\tilde{X}$ by $\tilde{(T)}(f)(x) = f(T(x))$. Allowing more general spaces $\tilde{X}$ C^* algebras allows the study of quantum versions. Also measure theoretical systems (X, T, μ) can be reformulated in function space. Instead of$\tilde{T}(f) = f(T)$ on all bounded measurable functions, consider the dynamics of a general unitary operator or more generally an automorphism on a von Neuman algebra. Also geometric structures have been "quantized" leading to a subject called "noncommutative geometry". The topic of **perturbation theory**, which is used for example to prove the persistence of stable motion (KAM) or the existence of homoclinic points (Melnikov theory). Finally, there is **spectral theory**, the study of the unitary operator $U_t f = f(T_t)$ on $L^2(X, \mu)$ for a map or flow T preserving a measure μ.

KNOWLEDGE VERSUS CREATIVITY. Even special areas of dynamical system theory have fragmented. It is relatively easy to be creative, when ignoring knowledge. It is much harder to find new results in the context of what is known. The right balance has to be found. In a first stage of research, avoiding the literature might be a good idea since too much information can be deadly for creative work. But after having figured out a way to solve the problem, looking up the literature is a necessity to face the possibility that a result has been proven already, maybe a special case of a much more general result. In that "library stage", a lot of information has to be processed in a short time. In a time, where patent offices pass sometimes requests which have a long time been "prior art" and in the public domain, some effort to pass some of the information which is available in books, in databases or papers to the brain has been made. Fortunately, technology softens some of the need to know vast amount of information. Still, most information is not online, nor in text books, not even in recent papers. The challenge is to balance two different but equally important things:

Acquire: process, absorb and learn information

Inquire: question, generate new ideas and solutions

HOMEWORK: most homework questions seemed have been just at the right level of difficulty a few were a notch too hard. I think most of the homework problems could not be solved without spending a few hours each week.

QUIZZES: the weekly quizzes tested knowledge and presence in the classroom. They also served as a tool to gauge, how the information have been absorbed during lecture.

PROJECTS: Most papers had a high standard. The topics ranged from history of dynamical systems to summaries. There was one project, which dealt with an unsolved problme: The exterior billiard project confirmed empirical evidence that the semicircle is an unstable billiard. Here are some project titles chosen for the end project:

The evolution of a universal grammar in the case of super symmetry	differential equations
HIV and Immune Respose Dynamics	differential equations
A Survey of Results Concerning the Collatz $3x + 1$ conjecture	discrete dynamics
Examples of Experimental Mathematics	Number theory
Exterior billiards on the Semicircle	2D maps
Continuous Newtonis Method	differential equations
Gravity in One Dimension	n-body problems
Stability in lineary stochastic systems	summary
A Survey of the History of Celestial Mechanics from Aristotle to Poincare	3 body problem
A dynamical systems view of the leaders dilemma	game theory