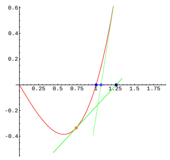
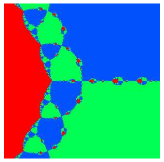
COMPLEX DYNAMICS — Math118, O. Knill

ABSTRACT. When maps are iterated in the complex plane it leads to interesting dynamics. An example is the Newton method in the complex. We look at some examples and especially show finally that the Ulam map is chaotic. Actually, the interval on which the Ulam map is defined is the Julia set of the corresponding quadratic map.

THE NEWTON METHOD IN THE REAL. The Newton method to find a root of $f(x) = 0$, is to start with a point x_0 and apply the map $T(x) = x - f(x)/f'(x)$. If $T(x) = x$, then $f(x) = 0$. Because $T'(x) = f(x)f''(x)/(f'(x))^2$ is small near $f(x) = 0$, T is a contraction in an interval $[x_0 - \epsilon, x_0 + \epsilon]$ and has a fixed point. The **basin of attraction** of a root x_i are all the points for which $T^n(x) \to x_i$.



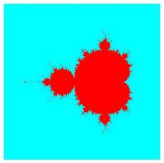
THE NEWTON METHOD IN THE COMPLEX. The Newton method to find a root $f(z) = 0$ can also be done in the complex plane. We start with a point z_0 and apply the map $T(z) = z - f(z)/f'(z)$. If $T(z) = z$, then $f(z) = 0$. Again $T'(z) = f(z)f''(z)/(f'(z))^2$ is small near $f(z) = 0$, the map T is a contraction. The **basin of attraction** of a root x_i are all the points for which $T^n(x) \to x_i$. The picture to the right shows the basins of attractions for each fixed point. Each of this region is the "stable manifold" of the fixed point. The rest is called the **Julia set** of T.



QUADRATIC MAP. The **quadratic map**

$$f_c : z \mapsto z^2 + c$$

with a complex parameter c defines s discrete dynamical system on the complex plane. f_c leaves a set $J_c \subset C$ called **Julia set** and its complement F_c, called the **Fatou set** invariant. The parameter space C is divided into a **Mandelbrot set** M, parameters, where J_c is connected and its complement, where J_c is disconnected.



PARAMETRIZING ALL QUADRATIC MAPS. The quadratic family f_x is not as special as one might think:

> LEMMA. A quadratic polynomial $T(z) = az^2 + 2bz + d$ is conjugated by $S(z) = az + b$ to
> $$f_c(z) = z^2 + c$$
> where $c = ad + b - b^2$.

Proof. Just verify $S^{-1} f_c S(z) = T(z)$.

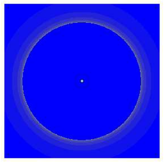
Remark. You show in the homework taht every cubic polynomial $T(z)$ can be conjugated to $f_{a,b}(z) = z^3 - 3a^2 z + b$. The parametrization is chosen so that $-a, a$ are critical points of $f_{a,b}$.
When dealing with maps on the real line, we could also chose the normal form

$$z \mapsto az(1-z) .$$

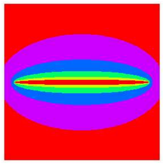
Parametrized like this, the quadratic map is also called the **logistic map**. It maps the interval $[0, 1]$ onto itself. The linear map $S(z) = -az + a/2$ conjugates $z \mapsto az(1-z)$ to $z \mapsto z^2 + c$, when $c = a/2 - a^2/4$. Especially, the Ulam map is conjugated to f_{-2}.

EXAMPLE. THE SQUARING MAP. Let us look at the map $f(z) = z^2$. If $z = re^{i\theta}$ with $r = |z|$, then $f^n(z) = r^{2^n} e^{i2^n\theta}$. If $r > 1$, then $f^n(z) \to \infty$. If $|r| < 1$, then $f^n(z) \to 0$. If $r = 1$, then $f^n(z) = e^{i2^n\theta}$. On $|z| = 1$, the map is $T(x) = 2x \bmod 1$.

> There is a set J on which f is chaotic and the complement F where f is attracted to some attracting fixed point.



EXAMPLE. THE ULAM MAP AS A QUADRATIC MAP. What happens with the Ulam map $f(z) = 4z(1-z)$ in the complex plane? We have seen that it is conjugated to $f_2(z) = z^2 - 2$. The conjugating map $S(z) = 2 - 4z$ maps the interval $[0, 1]$ to the interval $[-2, 2]$. This interval is invariant and the map T restricted to this interval is the Ulam map.

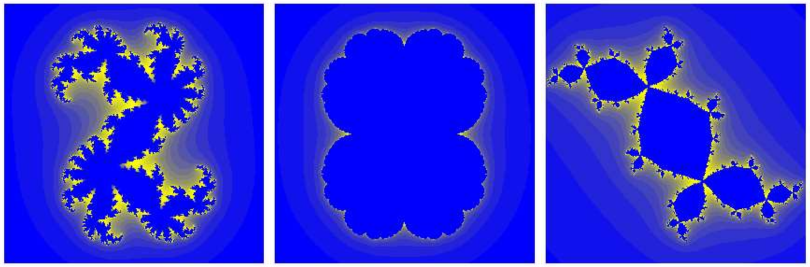


FIXED POINTS. The fixed points of the quadratic map are $z_\pm = (1 \pm \sqrt{1-4c})/2$. The value of $f'(z)$ determines the stability. If $|f'(z)| < 1$, then the fixed point is stable, if $|f'(z)| > 1$, it is **unstable**.

Note that when a complex map is written as a real map, then it is not possible that T has a hyperbolic fixed point.

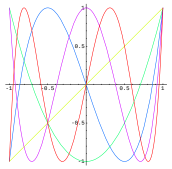
EXAMPLE. $f(z) = z^2 + z + 1$ has the fixed points $i, -i$. Since $f'(i) = 2i$ and $f'(-i) = -2i$, we have $|f'(i)| = 2$ and both fixed points are unstable.

JULIA SETS. Let f be a polynomial. Let P_c denote the set of all points for which $f^n(z)$ stays bounded. This is called the **prisoner set** K (or **filled in Julia set**). The boundary of K is called the **Julia set** J. The complement of J is an open set called the **Fatou set** F of f. It is known that the Julia set is the closure of all repelling periodic points. For the quadratic family, the Julia set is totally disconnected if c is outside the Mandelbrot set and connected, if c is inside the Mandelbrot set.



CHEBYSHEV POLYNOMIALS.

Let $f(z) = 2z^2 - 1$. Because $f(\cos(z)) = 2\cos^2(z) - 1 = \cos(2z)$ we have $f^n(\cos(z)) = \cos(2^n z)$. Actually, the map $S(z) = (z + 1/z)/2$ satisfies $Sz^2 = fS(z)$. In other words, the map S semiconjugates f to the map $g(z) = z^2$ which we have seen above. The conjugating map S maps the unit circle to the interval $[-1, 1]$. This can be used to conjugate the Ulam map to a shift. One generalize this example to the case, where $T_k(z)$ is the Chebychev polynomial $\cos(kz) = T_k(\cos(z))$. (See Homework).



THE ULAM MAP AND THE SHIFT.

> The Ulam map $T(x) = 4x(1-x)$ is chaotic in the sense of Devaney.

Proof. The Ulam map is conjugated to the Chebychev map $C(z) = 2z^2 - 1$. The idea is to use the semiconjugation of the later to $f(z) = z^2$ which is semiconjugated to the shift on $\{0, 1\}^N$. That the later is chaotic in the sense of Devaney had been shown last week in the CA week.

We can find $C(z)$, by forming $\theta = \arccos(z)$ and then get $y = \cos(2\theta)$. if $\arccos(z)/\pi = 0.x_1x_2x_3...$ in binary expansion, then $C(z) = \cos(\pi \cdot 0.x_2x_3x_4...)$.

To find a dense set of periodic points, take a periodic sequence $x \in \{0, 1\}^N$ then $z = \cos(\pi \cdot 0.x_1x_2...)$ is a periodic point of the Ulam map. The map $x \to z$ is continuous and surjectiv. We can find so periodic orbits intersecting each interval $[a, b]$. To show transitivity, take $z = \cos(\pi 0.x_1x_2...)$, a sequence $x \in\in \{0, 1\}^N$ which is transitive (concat an enumeration of all finite words onto each other)