CELLULAR AUTOMATA — Math118, O. Knill

ABSTRACT. A shift invariant continuous map on the sequence space A^Z over a finite alphabet A is called a **cellular automaton** or short a CA. These dynamical systems can be considered as discretized cousins of differential equations, for which time, space, as well as the configuration space are discretized.

THE NAME CELLULAR AUTOMATON. Interactions between different scientific fields is always productive. Historically, it seems that cellular automata were introduced in the late 40ies while some applied Mathematicians were dealing with problems from biology. The etymology of the name "CA" could confirm a "bonmot" of Stan Ulam:

> Ask not what mathematics can do for biology.
> Ask what biology can do for Mathematics.

Source: cited from David Campbell, who received his B.A. in chemistry and physics from Harvard in 1966 and worked in nonlinear science. Ulam himself was at Harvard from 1936-1939, eating at Adams house where "the lunches were particularly agreeable" and was also teaching the Math1A here (Source: Ulam: Adventures of a mathematician).
Anyway, it would not surprise if "cellular automaton" had been derived from "cellular spaces" because of mathematical research on biological problems.

SEQUENCE SPACES. Let A be a finite set called the **alphabet** and let A^Z denote the set of all sequences and $\sigma(x)_n = x_{n+1}$ the shift on X. A distance between two sequences is given by $d(x,y) = 1/(n+1)$, where n is the largest number such that $x_i = y_i$ for $|i| \leq n$. Example: Let $A = \{1,2,3,4\}$. For

...	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	...
...	1	1	4	3	2	1	1	...

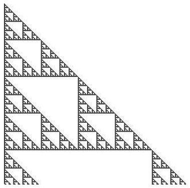
...	y_{-3}	y_{-2}	y_{-1}	y_0	y_1	y_2	y_3	...
...	1	2	3	3	4	1	1	...

we have $d(x,y) = 1/3$, because $x_i = y_i$ if $|i| \geq 3$ but $x_{-2} \neq y_{-2}$.

> LEMMA: X is a **compact metric space** (X,d).

PROOF. To have a metric space, show $d(x,x) = 0, d(x,z) \leq d(x,y) + d(y,z), d(x,y) = d(y,x)$. To have compactness, every sequence $x(k)$ in X must have an accumulation point. That is, there must exist a subsequence $x(k_l)$ in X which converges for $k \to \infty$. See homework.

1D-CELLULAR AUTOMATA. A continuous map T on X which commutes with σ is called a **cellular automaton**. A theorem of Curtis, Hedlund and Lyndon, which we will prove later implies that there is a function ϕ from $A^{2R+1} \to A$ such that $T(x)_i = \phi(x_{i-R}, x_{i-k+1}, ..., x_{i+R})$. The integer R is called the **radius** of the CA. It is assumed that R is the smallest number for which the CA still can be defined like that. One can visualize the dynamics of one dimensional CA by coding each letter in a sequence with a color. The first row is the initial condition. Applying the map gives the second row, etc. Drawing a few iterates produces a **phase space diagram**. The example shows the automaton over the alphabet $\{0,1\}$, where $x_n = x_n + x_{n-1}$ mod 2 and where 0 is black. If initially $x_n(0) = 0$ for $n \neq 0$ and $x_0(0) = 1$, we have an explicit solution formula with binomial coefficients $x_n(t) = \begin{pmatrix} n+t \\ n \end{pmatrix}$ mod(2).



CANTORS DIAGONAL ARGUMENT.

> THEOREM (Cantor) The set $X = A^Z$ is uncountable.

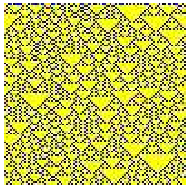
PROOF. If X were countable, one could enumerate all sequences $x(k)$ using integer indices k. Define the "Diagonal" sequence $y_n = (1 + x_n(|n|))$ (here $a+1$ is the next in the alphabet A, or the first element in A, if a was the last). The sequence y is different from any of the sequences $x(k)$ because y and $x(k)$ differ at the k'th entry. The assumption about the enumerability was not possible.

WOLFRAMS NUMBERING OF 1D CA. Any one-dimensional cellular automata with radius 1 and alphabet $\{0,1\}$ can be labeled by a **rule number**. Because there are $2^3 = 8$ possible maps ϕ, we have $2^8 = 256$ possible rules. The **Wolfram number** is $w = \sum_{k=1}^{8} f(k)2^k$, where $y_0 = f(k)$ is the new color for $k = 4x_{-1} + 2x_0 + x_1$.

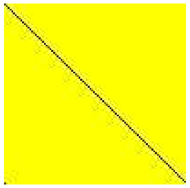
For example, let $\phi(a,b,c) = a$, then the new middle cell is 1 for the neighborhoods $111, 110, 101, 100$ which code the integers $7,6,5,4$. So, $f(7) = f(6) = f(5) = f(4) = 1$, and $f(k) = 0$ otherwise. The rule of the automaton is $w = 2^7 + 2^6 + 2^5 + 2^4 = 240$. Indeed, rule 240 is the shift automaton. Let us look at an other example.

EXAMPLES. The binominal CA discussed above has rule 90. One of the most studied CA is rule 18. Since $18 = 2^4 + 2^1$, which is 10010 to the base 2, we obtain the following function ϕ:

neighborhood (dec)	neighborhood (bin)	new middle cell	factor
7	111	0	128
6	110	0	64
5	101	0	32
4	100	1	16
3	011	0	8
2	010	0	4
1	001	1	2
0	000	0	1



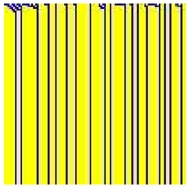
SPEED OF A CA. Every CA has a maximal speed c with which signals can propagate. This means if we take an initial conditions x which is constant outside an interval I, then then $T^k(x)$ will still be constant outside an interval I_k of size $|I_k| \leq |I||2c$



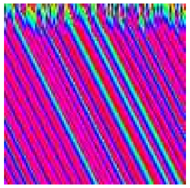
> LEMMA. The speed of a CA is bounded above by the radius R.

PROOF. Each timestep can change only cells maximally R units to the left or to the right.
Example: The **"Takahashi-Susama Soliton automaton"** is defined on points $x \in \{0,1\}^Z$ for which only finitely many cells are 1. The rule for T is to start from the left and move each 1 to the next 0 position. Since a pack of n adjecent $1's$ moves with speed n, the map T is **not** a cellular automaton.

EXAMPLES.
a) The cellular automaton $T = \sigma^c$ shifting $c \in \mathbf{N}$ entries to the right has the speed c. Since c is also the radius, this shows that the speed can not be faster then the radius R. The **speed ratio** c/R satisfies $c/R \leq 1$.
b) The CA $T(x)_n = (..., a, a, a, a, ...)$ is obtained by a function ϕ which is constant. Every orbit of this automaton is attracted to the fixed point. The speed is zero. The picture to the right shows rule-100 cellular automaton.



POSSIBLE SPEEDS. Note that we can enumerate the set of cellular automata: it is a countable set. Because the set of real numbers in the interval $[0,1]$ is uncountable, we can not obtain all the speeds.



> PROPOSITION. Fix A. For every $0 < a < b < 1$, there is a CA with radius R over the alphabet A for which the speed c satisfies $a \leq c/R \leq b$.

You explore this fact a bit in a homework. The idea is first to use a larger alphabet in order to slow down the motion using internal "color swapping". For different alphabets A, B, a A-automaton can be simulated by a B automaton, possibly changing the radius.