BILLIARDS I — Math118, O. Knill

ABSTRACT. The billiard dynamical system can be seen as a limiting case of a particle moving in the plane under the influence of a potential V. In the limit, the ODE of three variables becomes a simple map, which still has all the features of differential equations. We discribe the system as an extremization problem, show the existence of periodic orbits and the area-preservation property. We also see that the ellipse is an integrable billiard.

PARTICLE MOTION IN THE PLANE. The motion of a particle in the plane under the influence of a **force** $F(x,y) = (f(x,y), g(x,y) = -\nabla V(x,y)$ is described by the differential equations

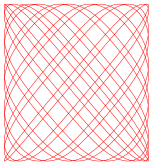
$$\begin{aligned} \frac{d^1}{dt^2}x(t) &= f(x,y) \\ \frac{d^2}{dt^2}y(t) &= g(x,y) \end{aligned}$$

Written as first order system, there are 4 variables x, y, u, v. Energy conservation $H(x,y,u,v) = u^2/2 + v^2/2 + V(x,y) = E$ reduces it to three variables:

$$\begin{aligned} \frac{d}{dt}x &= u \\ \frac{d}{dt}y &= \sqrt{2}\sqrt{E - V(x,y) - u^2/2} \\ \frac{d}{dt}u &= f(x,y) \end{aligned}$$

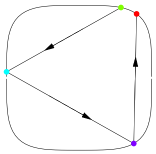
EXAMPLE. For $V(x,y) = x^4 + y^4$, the differential equations are

$$\begin{aligned} \frac{d}{dt}x &= u \\ \frac{d}{dt}y &= \sqrt{2}\sqrt{E - x^4 - y^4 - u^2/2} \\ \frac{d}{dt}u &= -4x^3 \end{aligned}$$

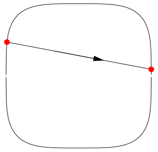


The picture shows an orbit close to a periodic orbit.

THE BILLIARD FLOW. Now, we take a particle in the plane and use a potential V which is zero inside a region G and which is infinite outside G. The mass point will move freely on a straight line until it hits the "wall". There it will reflect, bouncing off using the reflection law "incoming angle"="outgoing angle". The **Birkhoff billiard** is the dynamics of this billiard dynamical system, if the table is convex.



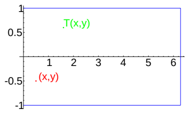
THE BILLIARD MAP. With an initial position s on the boundary, and an angle θ we have new initial position and a new angle. If the boundary of the table is parametrized by $x \in [0,1]$ and the angle by $\theta \in [0,\pi]$, we obtain a map $(s,\theta) \to (s_1,\theta_1)$.



BETTER COORDINATES. If we scale the table such that the table has length 1 and reparametrize the boundary of the table such that x is the **arc length** from some point 0 on the curve to s and take $y = \cos(\theta)$, we obtain a map

$$T : R/Z \times [-1,1],\ T(x,y) = (x_1,y_1)$$

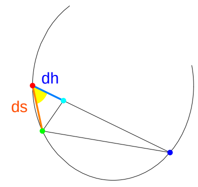
Topologically $R/Z \times [-1,1]$ is an annulus or a cylinder with boundary.

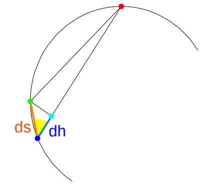


MONOTONE TWIST MAP. One boundary $R/Z \times \{-1\}$ is fixed and the other boundary $R/Z \times \{1\}$ is rotated once. Both boundaries, when the angle is 0 or π consist of fixed points. The map has the **twist property**: $\frac{d}{dy}x_1(x,y) > 0$. We prefer the (x,y) coordinates over the (s,θ) coordinates, because T becomes so area-preserving, as we will see below.

THE LENGTH FUNCTIONAL. Let $h(x_i, x_{i+1})$ denote the Euclidean distance between two points of the table (this is the distance in the plane and not the distance along the boundary). If $x_1, x_2, ..., x_n$ are successive impact points of the trajectory, then $\boxed{\cos(\theta_i) = -h_{x_i}(x_i, x_{i+1}) = h_{x_i}(x_{i-1}, x_i)}$

PROOF: You can see the relation $\cos(\theta) = dh/ds$ by watching the length change $dh = dh(x_i, x_{i+1})$, when x_i is replaced by $x_i + ds$ (first picture). The second formula is seen when observing the length change $dh = dh(x_{i-1}, x_i)$ when x_i is replaced with $x_i + ds$ (second picture).





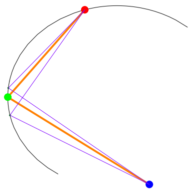
THE EULER EQUATIONS. The billiard map can be described by the equation

$$\boxed{h_{x_i}(x_i, x_{i+1}) + h_{x_i}(x_{i-1}, x_i) = 0}$$

This second order difference equation for the variables x_i is called the **Euler equation** of the billiard system. Given x_0, x_1, we can use these equations to get x_2, then use these equations again to get x_3 etc.

VARIATIONAL PRINCIPLE. If $x_1, x_2, ..., x_n$ is a sequence of impact points of the billiard map and the initial point x_0 and the final point x_{n+1} are fixed, then $x_0, x_1, x_2, ..., x_n$ is a billiard orbit if and only if $(x_1, x_2, .., x_{n-1})$ is a critical point of the function

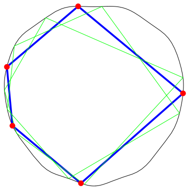
$$\boxed{H(x_1, x_2, ..., x_{n-1}) = \sum_{i=0}^{n} h(x_i, x_{i+1}).}$$



PROOF: just check that $\nabla H = 0$ gives the Euler equations. In other words, the billiard path extremizes the total length of the path. For $n = 2$, where we extremize $h(x_0, x_1) + h(x_1, x_2)$ we have to find the point x_1 on the table such that the path initiating at x_0 and ending at x_2 and which hits the table at a point x_1 is extremal.
This generalizes the Fermat principle: a light ray reflecting at a curve extremizes the distance to the curve only if in- and out-going angles are the same.

PERIODIC POINTS. A sequence $x_1, x_2, ..., x_n, x_{n=1} = x1$ is a periodic orbit if and only if the total length of the polygon of the impact points is extremal. In other words, we look for critical points of the total length of the closed polygon, which is:

$$\begin{aligned} H(x_1, ..., x_n) &= \sum_{i=1}^{n} h(x_i, x_{i+1}) \\ &= h(x_1, x_2) + h(x_2, x_3) + ... + h(x_{n-1}, x_n) + h(x_n, x_1) \end{aligned}$$



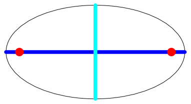
EXISTENCE OF PERIODIC POINTS. Since H is bounded, nonnegative and smooth, we have both a minimum and a maximum. The global minimum is of course when $x_1 = ..., x_n$ are all the same points. The maximum leads to a true periodic point: we have shown

> For a convex smooth billiard table, we find periodic points of minimal period n if n is prime.

PROOF. A continuous function on a bounded and closed subset of R^n has a maximum. The period can not be a factor of n because n was assumed to be prime. You show in a homework that the primality assumption is not necessary.

Example: The long axes and short axes of a convex table are periodic orbits of period 2.

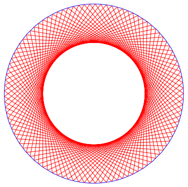
Example: Triangles of maximal total length in the table are billiard orbits of period 3.



BILLIARD IN A CIRCLE. The circle is an example of an integrable billiard. The angle θ and so $F(x, y) = y = \cos(\theta)$ is preserved. The billiard map T on $(R/Z) \times [-1, 1]$ is given explicitly by

$$T(x, y) = (x + 2\arccos(y)/(2\pi), y)$$

This is a shear map. On the first coordinate we have a **rational or irrational** rotation.

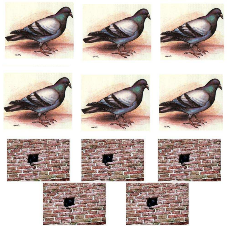


KRONECKER SYSTEM. The dynamical system on the circle obtained by a translation $T(x) = x + \alpha \bmod 1$ is called the **Kronecker system**. Let $x_n = [n\alpha] = n\alpha \bmod 1$ be the orbit of $T(x) = [x + \alpha]$ on the circle R/Z.

LEMMA. The sequence $x_n = T^n(x_0)$ is dense on $[0, 1]$ if α is irrational.

PROOF. Given n divide $[0, 1]$ into n equal intervals of length $1/n$. Take an orbit of length $n + 1$. By the **pigeon hole principle**, two of these points $0, \alpha, ..., n\alpha$ must be in the same interval and so have distance $< 1/n$. Therefore $\delta = m\alpha = (k - l)\alpha < 1/n$ for some integer m. With an integer N larger than $1/\delta$, the set $\{m\alpha = \delta, 2m\alpha = 2\delta, ..., mN\alpha = N\delta\}$ intersects every interval of length δ at least once. The set $\{x_0, x_1, ..., x_{mN}\}$ intersects every interval of length δ and so every interval of length $1/n$.

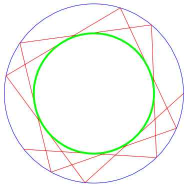
Illustration: 6 pigeons and 5 holes. Two pigeons must be in the same hole.



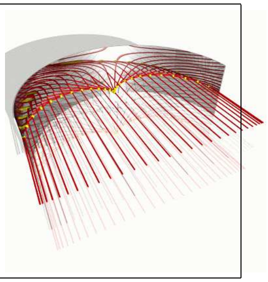
COROLLARY. If (s, θ) is an initial point for the billiard in a circle, then the orbit is periodic if $\theta/(2\pi)$ is rational. The ball will visit arbitrarily close to any given point of the table, if $\theta/(2\pi)$ is irrational.

CAUSTICS. For a billiard curve, one calls a curve a caustic, if the billiard ball, once tangent to that curve, remains tangent after the reflection.

EXAMPLE: For a circular table, every concentric circle inside the table is a caustic. For an ellipse, every confocal ellipse inside the table is a caustic.
EXAMPLE: given a convex curve, we can find a table which has this curve as a caustic using the **string construction**.



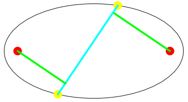
GENERAL CAUSTICS IN OPTICS. Places, where families of light rays focus are called **caustics**. If you take a family of parallel light and reflect it at a circle, then the light rays will focus at a curve which is called the **coffee cup caustic**. If the family of light rays is an orbit of a billiard ball in a table, then caustics might exist or not. In the case of the circle, every orbit produces caustics.



BILLIARD IN AN ELLIPSE.

The billiard in an ellipse is integrable.

PROOF. We find an invariant function $F(x, y)$, which is the product $d_1(x, y), d_2(x, y))$, where $d_i(x, y)$ is the distance of the trajectory to the focal point F_i. You will run a few lines of Mathematica to verify this in class.



BIRKHOFF-PORITSKY CONJECTURE: Is every integrable smooth convex billiard an ellipse? A collaborator of Birkhoff at Harvard with name **Hillel Poritsky** had worked on it and published a paper in 1950, where he made some progress.
The picture shows Poritsky in 1936 at the 42. Summer Meeting of the Mathematical Organizations of America in Cambridge, Massachusetts.

THEOREM. The billiard map is area-preserving.

PROOF. Let $Y \subset T^1 \times [-1, 1]$ be disc with boundary C. We show $\int\int_Y \, dydx = \int_Y \, dy'dx'$, where $T(x, y) = (x', y')$, $T^2(x, y) = (x'', y'')$ is the map. (We use primes here not as derivatives Using Greens formula, we get

$$\begin{aligned} \text{Area}(T^{-1}(Y)) &= \int\int_{T^{-1}(Y)} dy\, dx = \int_{T^{-1}(C)} y\, dx = \int_{T^{-1}(C)} h_1(x, x')\, dx \\ &= \int_C h_1(x', x'')\, dx' = \int_C -h_2(x, x')\, dx' = \int_C y'\, dx' = \int\int_Y dy'\, dx' = \text{Area}(Y) . \end{aligned}$$

GENERALIZATION. Every map defined by the Euler equations $h_2(x, x') + h_1(x', x'')$ of a smooth generating function $h(x, x')$ is area-preserving in the coordinates $(x, y) = (x, h_1(x, x')$.

EXAMPLE. $h(x, x') = (x' - x)^2/2 + V(x)$ leads to the Euler equation $h_1(x_i, x_{i+1}) + h_2(x_{i-1}, x_i) = (x_{i+1} - x_i) + \frac{d}{dx}V(x_i) - (x_i - x_{i-1}) =$ This is the second order difference equation $x_{i+1} - 2x_i + x_{i-1} + V'(x_i) = 0$. Vor $V(x) = c\cos(x)$, this recursion is the **Standard map**. For cubic V, it leads to the Henon map in the plane.

THE JACOBEAN MATRIX. An other proof to show that the map is area-preserving is to compute the Jacobean matrix and to verify that the determinant is 1. We will write down the Jacobean later. An other proof of the area-preservation property is given in proposition 6.4.2 of the textbook.

HISTORY.
Ludwig Boltzmann (1844-1906) studied the hard sphere gas. This is a billiard system.
Emil Artin (1898-1962) looked in 1924 at billiard in the hyperbolic plane. This is of interest in algebra.
Jacques Hadamard (1865-1963) Hedlund-Hopf studied the geodesic flow, which is a generalization of billiards.
George Birkhoff (1884-1944) in 1927, proposed convex billiards as a model for the 3-body problem
Hillel Poritsky in 1950 posed the integrability question.

WHY STUDY BILLIARDS?
It is a beautiful and simple dynamical system featuring all the complexities of more complex systems. It is a limiting case of the geodesic flow and illustrates theorems in topology, geometry or ergodic theory. It is related to Dirichlet spectral problem $\Delta u = \lambda u$ which can be considered the "quantum version" of the billiard problem, where the eigenfunctions describe a quantum particle moving freely in the table with energy λ.