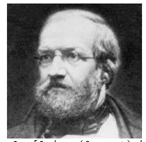
APPROXIMATION OF NUMBERS — Math118, O. Knill

ABSTRACT. The approximation of real numbers by rational numbers is a special and solvable case of solving the logarithm problem in dynamical systems.

DIRICHLET THEOREM. Let $x \in [0,1]$ be a real number in and $n > 1$ be an integer. There exist integers p and $1 \leq q \leq n$ $1 \leq p \leq n$ such that

$$|x - \frac{p}{q}| \leq \frac{1}{qn} .$$



PROOF. The Pigeonhole principle shows that at least one of the n intervals $[k/n,(k+1)/n]$ in $[0,1]$ contains two elements of the set $\{0,\{x\},\{2x\},...\{nx\}\}$, where $\{kx\}$ is the fractional part of kx. So $|(kx-lx)+p| \leq 1/n$ for some integer p and $q = k-l < n$. Division through q gives $|x+p/q| \leq 1/(nq)$.

APPROXIMATION. For any irrational x, there are infinitely many p/q such that $|x-p/q| \leq 1/q^2$.

PROOF. If x is rational, $q=0$ is possible and the result is not true. If x is irrational, then $k-l=0$ is not possible and $q>1$. Now $|x+p/q| \leq 1/(nq) \leq 1/q^2$.

CONTINUED FRACTION EXPANSION. We have seen the same result using continued fraction expansion p_n/q_n because

$$|x+p_n/q_n| \leq 1/(q_{n-1}q_n) \leq 1/q_n^2$$

There is a huge difference between this result and the above result

The pigeonhole principle is **not constructive**. It does not tell you what p/q is. The continued fraction expansion is **constructive**. You can determine p/q efficiently. The Dirichlet method needs n computations to determine the approximation, the continued fraction method essentially $\log(n)$.

THE LOG PROBLEM IN DYNAMICAL SYSTEMS.

Given a point x and a set I. At which time does the orbit of x enter I. For differential equations, we want to solve $T^t(x) = y$ up to some error, for maps, we want to solve $T^n(x) = y$ up to some error.

EXAMPLES.

- If $T(x) = x + \alpha$ mod1 and $x = 0$ is a real and $y = 0$. Determining $T^q(t) = 0$ is the problem to find n such that $|q\alpha - p| = y$ for some integer p. In other words, we want to find close solutions of $|\alpha - p/q| = 0$. The continued fraction expansion gives such values.
- The differential equation $\dot{x} = ax$ has the solution $T^t(x) = e^t x(0)$. To solve $T^t(x) = a^t = y$, we have $t = \log_a(y)$. Computation of the real logarithm is a special case of the dynamical logarithm problem.
- Given an prime number p and an integer a, we have a map $T(x) = ax$ mod p on the set $X = \{1,...p-1\}$. For given x and y, to compute n such that $T^n(x) = y$ is called the **discrete logarithm problem** in number theory. Logarithms are called **indices** in number theory. For a composite $n = pq$, if you could solve $a^k = 1$ mod n we could find p. For example $5^4 = 1$ mod 15 so that $\gcd(4+1,15) = 5$ is a factor. **The discrete log problem is harder then factoring**.
- If $T^t(x)$ is the evolution of the weather and x is the current meteorological condition and y is a severe storm, determining t such that $T^t(x)$ is close to y is a an example of a dynamical logarithm problem.
- If $T^t(x)$ is the position of an asteroid relatively to the earth and $y = 0$, then $T^t(x) = y$ determines the time it takes until the asteroid has an impact. It is an example of a dynamical logarithm problem.
- If T is the cellular automaton realization of a Turing machine, x is the initial condition with the empty tape and y is the "halt" state, then $T^n(x) = y$ determines how long it takes until the Turing machine halts. It is an example of a dynamical logarithm problem.

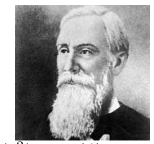
HURWITZ THEOREM. For any irrational x, there are infinitely many p/q such that $|x - \frac{p}{q}| \leq \frac{1}{\sqrt{5}q^2}$.

PROOF (Borel) One of the consecutive continued fraction convergent $p_{n-1}/q_{n-1}, p_n/q_n, p_{n+1}/q_{n+1}$ satisfies this bound. This is not so difficult to prove but could be part of a project.

This result can not be improved. The golden ratio satisfies this bound. There is an interesting story attached. If one takes away the bad example (the golden ratio) and all numbers which can be obtained by applying a modular transformation $T(x) = (ax+b)/(cx+d)$ with integers a,b,c,d satisfying $ad-bc=1$, then the bound $\sqrt{5}$ can be improved to $\sqrt{8}$ which is the best possible bound attained by the **silver ratio** $\sqrt{2}+1$.

SOLVING THE LOG PROBLEM FOR IRRATIONAL ROTATION. The following theorem solves the dynamical log problem for irrational rotations on the circle. Given two points on the circle, we can **construct** integers q_n such hat $T^n(x) = x + q_n\alpha$ is close to y.

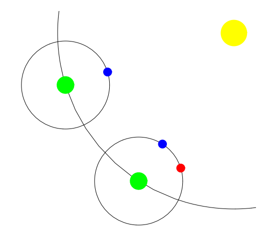
TCHEBYCHEV THEOREM. Assume x is irrational with periodic approximation p_n/q_n. Assume y is real. For every n, there exists $k \leq q_n$ such that $\{y+kx\} < 3/q_n$.



PROOF. Because $|x - p_n/q_n| \leq 1/(q_n q_{n-1})$, we can write $x = p_n/q_n + \delta/(q_n^2)$ with $|\delta| < 1$, where p_n/q_n are the periodic approximations of α.

Choose an integer t with $|q_n x - t| \leq 1/2$ so that $y = t/q_n + \delta'/(2q_n)$, $|\delta'| \leq 1$. Find k, l satisfying $q_n/2 \leq k \leq 3q_n/2$ with $p_n k - q_n l = t$. Then $|xk - l - y| = p_n k/q_n + \delta k/(q_n^2) - l - t/q_n - \delta'/(2q_n)| = |k\delta/q_n^2 - \delta'/(2q_n)| < k/(q_n q_n) + 1/(2q_n)$. Because $k < 3q_n/2$, the right hand side is $\leq 3/q_n$.

ECLIPSES AND PERIODIC APPROXIMATION. A **synodic month** is defined as the period of time between two new moons. It is $\alpha = 29.530588853$ days. The **draconic month** is the period of time to the moon to return to the same node. It is $\beta = 27.212220817$ days. Intersections between the path of the moon and the sun are called **ascending and descending nodes**. Such an intersection is called a solar eclipse. This appears in a period of a bit more then 18 years = 6580 days which is called one Saros cycle). This cycle and others are obtained from the continued fraction expansion of α/β. It is said that Thales using the Saros cycle to predict the solar eclipse of 585 B.C. The next big eclipse will happen May 26, 2021. Source: http://www.websters-online-dictionary.org/definition/english/mo/month.html



The Eclipse cycles can be explained using the continued fraction expansion (see homework).

cycle	eclipse	synodic	draconic
fortnight	14.77	0.5	0.543
month	29.53	1	1.085
semester	177.18	6	6.511
lunar year	354.37	12	13.022
octon	1387.94	47	51.004
tritos	3986.63	135	146.501
saros	6585.32	223	241.999
Metonic cycle	6939.69	235	255.021
inex	10571.95	358	388.500
exeligmos	19755.96	669	725.996
Hipparchos	126007.02	4267	4630.531
Babylonian	161177.95	5458	5922.999

See http://www.phys.uu.nl/ vgent/calendar/eclipsecycles.htm for more details.