PERIODIC POINTS AND LYAPUNOV EXPONENT — Math118, O. Knill

ABSTRACT. After distinguishing different types of periodic orbits of two dimensional maps, we look at the possible nature of periodic points and distinguish between elliptic, parabolic and hyperbolic cases, sources and sinks. We further introduce the Lyapunov exponent.

PERIODIC POINTS AND LINEARIZATION. Fixed points of the map T in the plane are called periodic points of period 1. Fixed points of T^n are periodic points of period n.

The Jacobean $DT(x,y) = T'(x,y)$ of a fixed point (x_0,y_0) plays an important role. It defines a linear map A which is called the **linearization** of T at the fixed point.

THEOREM. Near a fixed point (x_0,y_0), the map $T(x,y)-(x_0,y_0)$ is close to $DT(x_0,y_0)(x-x_0,y-y_0)$.

PROOF. The functions $f(x,y)$ and $g(x,y)$ have a Taylor expansion like $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x,y)(y-y_0) + f_{xx}(x_0,y_0)(x-x_0)^2/2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2/2 + \dots$. Terms like $(x-x_0)^2$ are small near the fixed point (x_0,y_0).

It follows that that if we iterate only for a fixed number of points, we can approximate the real map with the linearized map. However, because orbits will in general move away from the fixed point, where the linearization will no more be a good approximation, we can not expect a global correspondence. We will see that under some conditions, we can deduce something from the knowledge of the linearization.

EXAMPLE: THE STANDARD MAP. The map $T(x,y) = (2x + c\sin(x) - y, x)$ is a map on the plane. It can also be considered a map on the torus because $T(x+2\pi,y) = T(x,y) + (4\pi,2\pi)$, $T(x,y+2\pi) = T(x,y) + (-2\pi,0)$. The map is called the Standard map. The Jacobean matrix at a point (x,y) is

$$DT(x,y) = T'(x,y) = \begin{bmatrix} 2+c\cos(x) & -1 \\ 1 & 0 \end{bmatrix} .$$

Because the determinant of the Jacobean is 1 at all points, the map is area-preserving for all parameters c.

At the fixed point $(0,0)$, the Jacobean matrix is

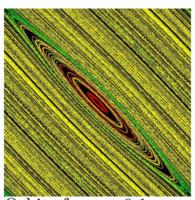
$$T'(0,0) = \begin{bmatrix} 2+c & -1 \\ 1 & 0 \end{bmatrix}$$

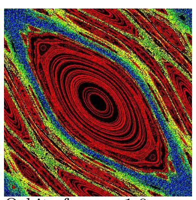
The eigenvalues are real and different for $c > 0$.

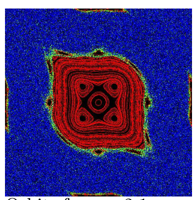
At the fixed point (π,π), the Jacobean matrix is

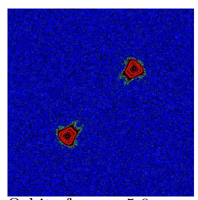
$$T'(\pi,\pi) = \begin{bmatrix} 2-c & -1 \\ 1 & 0 \end{bmatrix}$$

λ_i imaginary for $0 < c < 4$ and real for $c > 4$.

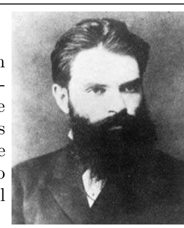

Orbits for $c = 0.1$.


Orbits for $c = 1.0$.


Orbits for $c = 2.1$.


Orbits for $c = 5.0$.

THE STABILITY QUESTION.

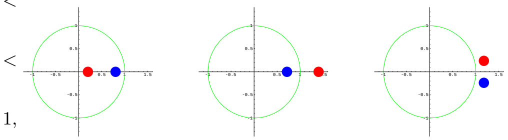


For nonlinear dynamical systems, the question of stability of fixed points can be very difficult. A pioneer in stability theory was Aleksandr Lyapunov (1857-1918). It turns out that already for simple cases like the Henon map or the Standard map, the stability of points, where the linearization is a rotation is difficult to establish. In the case, when the eigenvalues are real and both have not absolute value 1, then one can conjugate the map near the fixed point to its linearization. In those cases, the linearized picture essentially gives the real picture near the fixed point.

EIGENVALUES OF LINEAR MAPS. The characteristic polynomial of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$. If $c \neq 0$, the eigenvalues are $\lambda_\pm = \mathrm{tr}(A)/2 \pm \sqrt{(\mathrm{tr}(A)/2)^2 - \det(A)}$.

TYPICAL FIXED POINTS. If $T(x,y)$ is a differentiable map and $T(x_0,y_0) = (x_0,y_0)$ is fixed point with Jacobean $A = DT(x_0,y_0)$. Using the eigenvalues λ_1, λ_2 of A, we define the following **typical cases**, typical in the sense that the property is stable under small changes of parameters of the map:

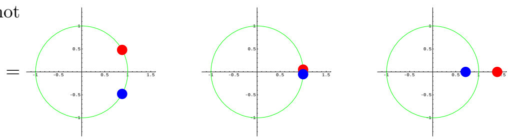
- **hyperbolic sink** $|\lambda_1| < 1, |\lambda_2| < 1$.
- **hyperbolic saddle** $|\lambda_1| < 1 |\lambda_2| > 1$
- **hyperbolic source** $|\lambda_1| > 1$, $|\lambda_2| > 1$.



EXAMPLE. Fixed points of the quadratic Henon map $T(x,y) = (1 - ax^2 - y, bx)$ are of the form (x, bx). Lets look at the case $a = 1.4, b = 0.3$. Solving $1 - ax^2 - bx = x$ gives the fixed points $(-10/7, -3/7)$ and $(1/2, 3/20)$. At the fixed point $(-10/7,-3/7)$ the eigenvalues are $\lambda_1 = (20 + \sqrt{370})/10 = 3.92..$ and $\lambda_2 = (20 - \sqrt{370})/10 = 0.07646...$ At the fixed point $(1/2, 3/20)$ the eigenvalues are $\lambda_1 = (-7 - \sqrt{19})/10 = -1.13589, \lambda_2 = (-7 + \sqrt{19})/10 = -0.26411$. We see that both fixed points are hyperbolic.

TYPICAL FIXED POINTS OF AREA-PRESERVING MAPS. If $\det(DT(x,y)) = 1$ for all (x,y) then T is area-preserving by the change of variable formula. In that case $\lambda_1\lambda_2 = 1$ and sinks or sources are no more possible. Cases with $|\lambda_i| = 1$ can now persist under parameter changes, if the deformation happens in the class of area-preserving maps. We distinguish now between the following cases:

- **elliptic** $|\lambda_1| = |\lambda_2| = 1$, λ_i not real.
- **parabolic** $\lambda_1 = \lambda_2 = -1$ or $\lambda_1 = \lambda_2 = 1$.
- **hyperbolic** $|\lambda_1| < 1 |\lambda_2| > 1$



Parameter values, for which a periodic orbit changes from hyperbolic to elliptic or in the other direction are called **bifurcation parameters**.

THEOREM. A fixed point of an area preserving map is elliptic if $|\mathrm{tr}(DT)| < 2$. It is parabolic if $|\mathrm{tr}(DT)| = 2$ and hyperbolic, if $|\mathrm{tr}(DT)| > 2$.

PROOF. Distinguish between the cases $\det(DT) = 1$ and $\det(DT) = -1$.

THE NORM OF A MATRIX. For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ define the **norm** $||A|| = \sqrt{tr(AA^T)} = \sqrt{a^2+b^2+c^2+d^2}$. Remember that A^T is the **transpose** of the matrix and $tr(A)$ denotes the **trace** of a matrix A.

Side remark. There are different ways to define the norm. The usual norm $||A|| = \max_{|v|=1}|Av|$ is known to be the square root of the largest eigenvalue of A^TA but is less convenient to compute.

LYAPUNOV EXPONENT. The exponential growth rate of $||DT^n(x,y)||$ is

$$\lambda(T,(x,y)) = \liminf_{n\to\infty} \frac{1}{n} \log ||DT^n(x,y)||$$

is called the **Lyapunov exponent** of T at the point (x,y). For area preserving maps T on the torus, define $\lambda(T) = \lim_{n\to\infty} \frac{1}{n} \int\int \log ||DT^n(x)|| \, dxdy$ is known to be to a quantity called the **entropy** of the map.

Examples:
a) If $T(x,y) = (x+\alpha, y+\beta)$, then the Lyapunov exponent is zero for every orbit.
b) If $T(x,y) = (2x+y, x+y)$ is the cat map on the torus, then the Lyapunov exponent is $\log(|3+\sqrt{5}|/2)$ for all orbits
c) In the case of the Standard map, one does not know the Lyapunov exponent for most orbits. One numerically measures an entropy $\geq \log(c/2)$.