QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday January 27, 2009 (Day 1)

1. Let $\mathbb{P}^{n^2-1}$ be the space of nonzero $n \times n$ matrices mod scalars, and consider the subset

$$\Sigma \;=\; \{(A,B) : AB = 0\} \;\subset\; \mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1}.$$

(a) Prove that Σ is a Zariski closed subset of $\mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1}$.

(b) Is Σ irreducible?

(c) What is the dimension of Σ?

2. Consider the integral

$$\int_0^\infty \sin x \cdot x^{a-1} dx.$$

(a) For which real values of a does the integral converge absolutely? For which does it converge conditionally?

(b) Evaluate the integral for those values of a for which it does converge.

3. (a) Let p be a prime number. Show that a group G of order p^n $(n > 1)$ has a nontrivial normal subgroup, that is, G is not a simple group.

(b) Let p and q be primes, $p > q$. Show that a group G of order pq has a normal Sylow p-subgroup. If G has also a normal Sylow q-subgroup, show that G is cyclic.

(c) Give a necessary and sufficient condition on p and q for the existence of a non-abelian group of order pq. Justify your answer.

4. Let $X = S^1 \vee S^1$ be a figure 8.

(a) Exhibit two three-sheeted covering spaces $f : Y \to X$ and $g : Z \to X$ such that Y and Z are not homeomorphic.

(b) Exhibit two three-sheeted covering spaces $f : Y \to X$ and $g : Z \to X$ such that Y and Z are homeomorphic, but not as covering spaces of X (i.e., there is no homeomorphism $\phi : Y \to Z$ such that $g \circ \phi = f$).

(c) Exhibit a normal (that is, Galois) three-sheeted covering space of X.

(d) Exhibit a non-normal three-sheeted covering space of X.

(e) Which of the above would still be possible if we were considering two-sheeted covering spaces instead of three-sheeted?.

5. Suppose T is a bounded operator in a Hilbert space V and there exist a basis $\{e_k\}$ for V such that $Te_k = \lambda_k e_k$. Prove that T is compact if $\lambda_k \to 0$ as $k \to \infty$.

6. Let $\Sigma \subset \mathbb{R}^3$ be a smooth 2-dimensional submanifold, and $n : \Sigma \to \mathbb{R}^3$ a smooth map such that $n(p)$ is a unit length normal to Σ at p. Identify the tangent bundle $T\Sigma$ as the subspace of pairs $(p, v) \in \Sigma \times \mathbb{R}^3$ such that $v \cdot n(p) = 0$, where $\cdot$ designates the Euclidean inner product. Suppose now that $t \to p(t)$ is a smoothly parametrized curve in $\mathbb{R}^3$ that lies on Σ. Prove that this curve is a geodesic if and only if

$$p''(t) \cdot (n(p(t)) \times p'(t)) = 0 \quad \forall t$$

Here, p' is the derivative of the map $t \to p(t)$ and p'' is the second derivative.

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Wednesday January 28, 2009 (Day 2)

1. Let $C \subset \mathbb{P}^n$ be a smooth algebraic curve.

(a) Let $\Lambda \subset \mathbb{P}^n$ be a general $(n-4)$-plane. Show that the projection map $\pi_\Lambda : C \to \mathbb{P}^3$ is an embedding.

(b) Now let $\Lambda \subset \mathbb{P}^n$ be a general $(n-3)$-plane. Show that the projection map $\pi_\Lambda : C \to \mathbb{P}^2$ is birational onto its image, and that the image curve has only nodes (ordinary double points) as singularities.

2. Show that the function defined by

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

is analytic in the open disc $|z| < 1$, but has no analytic continuation to any larger domain.

3. (a) Let K be the splitting field of the polynomial $f(x) = x^3 - 2$ over $\mathbb{Q}$. Find the Galois group G of $K/\mathbb{Q}$ and describe its action on the roots of f.

(b) Let K be the splitting field of the polynomial $X^4 + aX^2 + b$ (where a, $b \in \mathbb{Q}$) over the rationals. Assuming that the polynomial is irreducible, prove that the Galois group G of the extension $K/\mathbb{Q}$ is either C_4, or $C_2 \times C_2$, or the dihedral group D_8.

4. Let $\{f_n\}$ be a sequence of functions on the interval $X = (0, 1) \subset \mathbb{R}$, and suppose $f_n \to f$ in $L_p(X)$ for all $p : 1 \leq p < \infty$. Does it imply that $f_n \to f$ almost everywhere? Does it imply that there is a subsequence of f_n converging to f almost everywhere? Prove your answer or give a counterexample.

5. View S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$, and in particular S^1 as the unit circle in $\mathbb{C}$. Define an action of S^1 on S^{2n+1} by

$$\lambda : (z_1, \ldots, z_{n+1} \mapsto (\lambda z_1, \ldots, \lambda z_{n+1})$$

The quotient is the space $\mathbb{CP}^n$. View the projection map $\pi : S^{2n+1} \to \mathbb{CP}^n$ as a principal S^1-bundle.

(a) Explain why the restriction to S^{2n+1} of the 1-form

$$A = \frac{1}{2} \sum_{1 \leq k \leq n=1} (\overline{z}_k dz_k - z_k d\overline{z}_k)$$

defines a connection on this bundle.

(b) What is the pullback to S^{2n+1} of the curvature 2-form of this connection?

6. Let $X = S^2 \times \mathbb{RP}^3$ and $Y = S^3 \times \mathbb{RP}^2$

(a) Find the homology groups $H_n(X, \mathbb{Z})$ and $H_n(Y, \mathbb{Z})$ for all n.

(b) Find the homology groups $H_n(X, \mathbb{Z}/2)$ and $H_n(Y, \mathbb{Z}/2)$ for all n.

(c) Find the homotopy groups $\pi_1(X)$ and $\pi_1(Y)$.

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Thursday January 29, 2009 (Day 3)

1. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be an algebraic curve of bidegree (a, b) (that is, the zero locus of a bihomogeneous polynomial of bidegree (a, b)), and let $C' \subset \mathbb{P}^3$ be the image of C under the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.

(a) What is the degree of C'?

(b) Assume now that $\max(a, b) \geq 3$. Show that C' lies on one and only one quadric surface $Q \subset \mathbb{P}^3$ (namely, the quadric surface $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$).

2. Find the Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

around 0 of the function

$$f(z) = \frac{1}{z^2 + z + 1}$$

(a) valid in the open unit disc $\{z : |z| < 1\}$, and

(b) valid in the complement $\{z : |z| > 1\}$ of the closed unit disc in $\mathbb{C}$.

3. Let A be a commutative ring. Show that an element $a \in A$ belongs to the intersection of all prime ideals in A if and only if it's nilpotent.

4. Let f be a given real-valued function on $X = (0, 1) \subset \mathbb{R}$, and define a function $\phi : [1, \infty) \to \mathbb{R}$ by

$$\phi(p) = \|f\|^p_{L^p(X)}.$$

Prove that ϕ is convex.

5. Let $X \subset \mathbb{R}^2$ be a connected one-dimensional real analytic submanifold, not contained in a line. Prove that not every tangent line to X is bitangent—that is, it is not the case that for all $p \in X$ there exists $q \neq p \in X$ such that the tangent line to X at p equals the tangent line to X at q as lines in $\mathbb{R}^2$.

6. Let X and Y be two CW complexes.

(a) Show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

(b) Let A and B be two subcomplexes of X such that $X = A \cup B$. Show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.