QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday 25 January 2005 (Day 1)

1. (a) Show that, up to isomorphism, there is a unique group of order 15.

(b) Show that, up to isomorphism, there are exactly two groups of order 10.

2. Let m and n be positive integers, and k another positive integer less than m and n. Let $N = mn - 1$, and realize the complex projective space $\mathbb{P}^N$ as the space of nonzero $m \times n$ complex matrices modulo scalars. Let $X_k \subset \mathbb{P}^N$ be the subset of matrices of rank k or less. Show that X_k is an irreducible closed algebraic subset of $\mathbb{P}^N$, and compute its dimension.

3. (a) Consider $f(x) \in L^1(\mathbb{R}^n)$, $\mathbb{R}^n$ equipped with Lebesgue measure. Show that the function

$$\Phi_f : \mathbb{R}^n \to L^1(\mathbb{R}^n) \text{ given by } \Phi_f(y)(x) = f(x+y)$$

is continuous.

(b) Let $f(x) \in L^1(\mathbb{R})$ and also absolutely continuous. Prove that

$$\lim_{h \to 0} \int \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| dx = 0.$$

4. Let $-1 < a < 1$ and $0 < \beta < \pi$. Compute

$$\int_0^\infty \frac{x^a \, dx}{1 + 2x\cos\beta + x^2}$$

and express the answer as a rational function of π, $\sin a\beta$, $\sin a\pi$, $\sin\beta$ with rational coefficients.

5. Let $f : [a, b] \to \mathbb{R}_+$ be smooth, and let $X \subset \mathbb{R}^3$ be the surface of revolution formed by revolving $z = f(x)$ about the x-axis.

(a) For which functions f is the Gaussian curvature of X

- always positive?
- always negative?
- identically zero?

(b) Characterise by a differential equation those functions f such that X is a minimal surface.

6. Illustrate how a Klein bottle, K, may be decomposed as the union of two Möbius bands, joined along their common boundary. Using this decomposition and van Kampen's theorem, obtain a presentation of $\pi_1(K)$, and hence show that there is surjection from $\pi_1(K)$ to the dihedral group of order $2n$, for all n.

Prove that, for all $n \geq 3$, there exists a connected n-sheeted covering space, $\tilde{K}_n \to K$, that is *not* a normal covering. What is the topology of the covering space $\tilde{K}_5$ in your example?

QUALIFYING EXAMINATION

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Department of Mathematics

Wednesday 26 January 2005 (Day 2)

1. (a) Show that every group of order p^n, p prime, has a nontrivial center.

(b) Let G be a group of order p^n, let k be a (possibly infinite) field of characteristic p, and let M be a finite-dimensional k-vector space on which G acts. Show that if $\sigma \in G$ satisfies $\sigma^p = 1$, then σ fixes (pointwise) a nonzero subspace of M.

(c) Under the same assumptions as in the previous part, show that there is a nonzero vector of M fixed by G.

2. Let H_1 and H_2 be two distinct hyperplanes in $\mathbb{P}^n$. Show that any regular function on $\mathbb{P}^n - (H_1 \cap H_2)$ is constant.

3. (a) A (formal) sum $\sum_{n=1}^{\infty} a_n$ of complex numbers $a_n \in \mathbb{C}$ is *Cesaro summable* provided the limit

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} S_i$$

exists, where $S_i := \sum_{j=1}^{i} a_j$. Given a sequence of complex numbers $\{a_n\}_{n \geq 1}$, assume $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$, and $\sum_{n=1}^{\infty} a_n$ is Cesaro summable. Show that $\sum_{n=1}^{\infty} a_n$ converges.

(b) Let $f(\theta) \in C^0(S^1)$ (a continuous function on the unit circle) with the property $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 |n| < \infty$, where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} \, d\theta.$$

Show that

$$S_n f \to f \text{ uniformly}$$

as $n \to \infty$ where $S_n f$ are the partial Fourier sums of f, i.e.

$$S_n f(\theta) = \sum_{k=-n}^{n} \hat{f}(k) e^{ik\theta}.$$

4. Let $f(z)$ be a holomorphic function on $\{|z| < 1\}$ and continuous up to $\{|z| \leq 1\}$. Let M be the supremum of $|f|$ on $\{|z| \leq 1\}$. Let L be the intersection of $\{|z| \leq 1\}$ and $\left\{\operatorname{Re} z = \frac{1}{2}\right\}$. Let m be the supremum of $|f|$ on L. Show that $|f(0)|^3 \leq m\, M^2$. (*Hint:* Consider the product of some functions obtained from f.)

5. Let C_1 and C_2 be smooth curves in $\mathbb{R}^3$. Suppose that there exist unit-speed parametrizations $\rho_1 : \mathbb{R} \to \mathbb{R}^3$, $\rho_2 : \mathbb{R} \to \mathbb{R}^3$ of C_1, C_2 such that:

- the curvatures κ_1, κ_2 of ρ_1, ρ_2 coincide and are never zero;
- the torsions τ_1, τ_2 of ρ_1, ρ_2 coincide.

Show that there exists an isometry I of $\mathbb{R}^3$ taking C_1 to C_2.

6. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$, and let $f : S^m \to S^n$ be a map satisfying $f(-x) = -f(x)$ for all x. Assuming that m and n are both at least 1, show that the resulting map $\bar{f} : \mathbb{RP}^m \to \mathbb{RP}^n$ induces a non-zero map $\bar{f}_* : \pi_1(\mathbb{RP}^m) \to \pi_1(\mathbb{RP}^n)$. Use $\mathbb{Z}/2$ cohomology to show that $m \leq n$.

In the case that n is even and $m = n$, show that $\bar{f}$ must have a fixed point.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday 27 January 2005 (Day 3)

1. Let p be a prime, and let K be a field of characteristic not equal to p that contains the pth roots of unity. Show that every cyclic extension L of K of degree p can be obtained by adjoining a root of the polynomial $x^p - a$ for some $a \in K$.

2. Let X be the Veronese surface, i.e. the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. If $C \subset X$ is a closed irreducible curve, show that there exists a hypersurface $H \subset \mathbb{P}^5$ such that $H \cap X = C$, where the intersection is considered set-theoretically.

3. Show that

$$\|f * g\|^2_{L^2(\mathbb{R})} \leq \|f * f\|_{L^2(\mathbb{R})} \|g * g\|_{L^2(\mathbb{R})}$$

for all $f, g \in L^2(\mathbb{R})$ (with the understanding that either side may be infinite). Can there be such an inequality with $L^1(\mathbb{R})$ instead of $L^2(\mathbb{R})$? (Hint: use delta functions.)

4. By applying the Argument Principle to the domain which is the intersection of a quadrant and the disk centered at the origin of radius R with $R \to \infty$, find out how many roots of the following equation lie in each of the four quadrants:

$$z^4 + z^3 + 4z^2 + 2z + 3 = 0.$$

(*Hint:* First observe that there are no zeroes on the nonnegative real axis and the imaginary axis. Then verify that there are no zeroes on the negative real axis by separately grouping certain terms together in the case $\operatorname{Re} x \leq -1$ and in the case $\operatorname{Re} x > -1$. Then consider the change of arguments along the positive imaginary axis and a large quarter-circle.)

5. Let ω be a closed non-degenerate 2-form on a compact smooth manifold M, and let $f : M \to \mathbb{R}$ be smooth.

(a) Show that there is a unique vector field X on M such that $\iota_X \omega = df$.

(b) For each $t \in \mathbb{R}$, let $\rho_t : M \to M$ be the time-t flow of the vector field X. Show that $\rho_t^\star \omega = \omega$ and $\rho_t^\star f = f$ for all $t \in \mathbb{R}$.

(c) Let M be the unit 2-sphere in $\mathbb{R}^3$ and let ω be the standard volume form on M. Find a function $f : M \to \mathbb{R}$ so that the corresponding map ρ_t is rotation about the z-axis by the angle t.

6. Give an example of a pair of CW complexes, (X, A), satisfying all of the following three conditions: (i) there exists an n such that X is obtained from A by adding a single n-cell,

$$X = A \cup_\phi e^n;$$

(ii) the attaching map $\phi : S^{n-1} \to A$ for this n-cell induces the zero map $H_{n-1}(S^{n-1}) \to H_{n-1}(A)$; and (iii) the space A is not a retract of X. Justify your answer.