QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday August 30, 2011 (Day 1)

1. Let f be a differentiable function on $\mathbb{R}$ whose Fourier transform is bounded and has compact support.

(a) Prove that there exists a constant $C \in \mathbb{R}$ such that the k^{th} derivative of f is bounded by C^{k+1} for all $k \geq 0$.

(b) Prove that f does not have compact support unless it is identically zero.

2. Let $F \subset K \subset L$ be fields.

(a) Show that $[L : F] = [L : K][K : F]$.

(b) Compute $[\mathbb{Q}(\sqrt{3}, \sqrt{2}) : \mathbb{Q}]$.

(c) Show that $x^3 - \sqrt{2}$ is irreducible over $\mathbb{Q}(\sqrt{2})$.

3. Consider the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\varphi(X, Y, Z) = (XY, YZ, XZ)$.

(a) Show that φ is birational.

(b) Find open subsets $U, V \subset \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.

(c) Let $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of φ (that is, the closure in $\mathbb{P}^2 \times \mathbb{P}^2$ of the graph of the map on any open set where it's regular). Describe the projection $\pi_1 : \Gamma \to \mathbb{P}^2$ as a blow-up of $\mathbb{P}^2$.

4. For any positive integer n, evaluate

$$\int_0^\infty \frac{x^{1/n}}{1 + x^2} dx$$

5. Let M be a closed manifold (compact, without boundary). Let $f : M \to \mathbb{R}$ be a smooth function. For $t \in \mathbb{R}$ let $X_t = f^{-1}(t)$. If there is no critical value of f in $[a, b]$ show that X_a and X_b are submanifolds, and X_b is diffeomorphic to X_a.

6. A covering space is *abelian* if it is normal and its group of deck transformations is abelian. Determine all connected abelian covering spaces of $S^1 \vee S^1$ (the figure 8). (Hint: one way to do this might be to consider their relation to covering spaces of $S^1 \times S^1$.)

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday August 31, 2011 (Day 2)

1. Find the Laurent expansion of the meromorphic function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

(a) valid in the open unit disc $|z| < 1$;

(b) valid in the annulus $1 < |z| < 2$; and

(c) valid in the complement $|z| > 2$ of the closed disc of radius 2 around 0.

2. For any closed, connected, compact, oriented n-manifolds X and Y, write $X\#Y$ for their oriented connected sum.

(a) Show that $H_i(X\#Y;\mathbb{Z}) \cong H_i(X;\mathbb{Z}) \oplus H_i(Y;\mathbb{Z})$ for all $0 < i < n$.

(b) Compute the cohomology ring $H^*((S^2 \times S^8)\#(S^4 \times S^6);\mathbb{Z})$ and show in particular that it satisfies Poincaré duality.

3. Let

$$S^3 = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\},$$

and let α be the 1-form on $\mathbb{R}^4$ given by

$$\alpha = xdy - ydx + zdt - tdz.$$

(a) Prove that the restriction of the form $\alpha \wedge d\alpha$ to S^3 is nowhere 0.

(b) Compute the integral of $\alpha \wedge d\alpha$ over S^3.

(c) Let $U \subset S^3$ be an open subset, and let v and w be everywhere independent vector fields on U with $\alpha(v) \equiv \alpha(w) \equiv 0$. If $[v, w]$ is the Lie bracket of v and w, show that $\alpha([v, w])$ is nowhere zero on U. (Hint: use polar coordinates on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$.)

4. Let $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2 + 1)$.

(a) What are the units in the ring $\mathbb{Z}[i]$?

(b) What are the primes in $\mathbb{Z}[i]$?

(c) Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

5. Let $Y \subset \mathbb{P}^n$ be an irreducible variety of dimension r and degree $d > 1$, and let $p \in Y$ be a non-singular point. Define X to be the closure of the union of all lines $\overline{p, q}$, with $q \in Y$ and $q \neq p$.

(a) Show that X is a variety of dimension $r+1$.

(b) Show that the degree of X is strictly less than d.

(c) Give an example where the degree of X is strictly less than $d-1$.

6. Let $L^2(\mathbb{R})$ denote the completion of the Banach space of smooth functions with compact support using the norm whose square is

$$\|f\|^2 = \int_{\mathbb{R}} f^2;$$

and let $L^2_1(\mathbb{R})$ be the completion of the Banach space of smooth functions with compact support using the norm whose square is

$$\|f\|_1^2 = \int_{\mathbb{R}} (\frac{df}{dx})^2 + f^2.$$

(a) Prove that the map $f \mapsto \frac{df}{dx}$ from the space of smooth, compactly supported functions to itself extends to a bounded, linear map ϕ from $L^2_1(\mathbb{R})$ to $L^2(\mathbb{R})$.

(b) Prove that this extended map ϕ does not have closed image.

(c) Prove that the map $f \mapsto \frac{df}{dx} - f$ from the space of smooth, compactly supported functions to itself extends to an isometry from $L^2_1(\mathbb{R})$ to $L^2(\mathbb{R})$.

(d) Prove that the map $f \mapsto \frac{df}{dx} - \frac{x}{\sqrt{1+x^2}} f$ has closed image and 1-dimensional cokernel.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 1, 2011 (Day 3)

1. Let Λ_1, Λ_2 and $\Lambda_3 \subset \mathbb{P}^{2n+1}$ be pairwise skew (that is, disjoint) n-planes, and let $X \subset \mathbb{P}^{2n+1}$ be the union of all lines $L \subset \mathbb{P}^{2n+1}$ that meet all three.

(a) Show that through every point $p \in \Lambda_1$ there is a unique line meeting both Λ_2 and Λ_3.

(b) Show that $X \subset \mathbb{P}^{2n+1}$ is a closed subvariety.

(c) What is the dimension of X?

2. (a) Define the *degree* of a map $f\colon S^n \to S^n$

(b) Show that the degree of f is zero if f is not surjective.

(c) Show that if f has no fixed points, it has the same degree as the antipodal map. What is this degree?

(d) Show that $\mathbb{Z}/2$ is the only group that can act freely on S^{2n}.

3. Let p be a prime and $G = \mathrm{GL}_2(\mathbb{F}_p)$.

(a) Find the order of G.

(b) Show that the order of every element of G divides either $(p^2 - 1)$ or $p(p-1)$.

4. Let $H = \{z = x + iy : y > 0\} \subset \mathbb{C}$ be the upper half plane, with the metric

$$ds^2 = \frac{dxdy}{y^2}.$$

(a) What are the equations for the geodesics? Prove that they are either straight lines or semicircles.

(b) Compute the scalar curvature.

5. Let X be a Banach space. Assume that the dual X^* of X is separable. Show that X is separable.

6. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous on all of $\mathbb{C}$ and analytic on $\mathbb{C} \setminus [-1, 1]$. Show that f is entire, that is, analytic on all of $\mathbb{C}$.