QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday October 2, 2001 (Day 1)

Each question is worth 10 points, and parts of questions are of equal weight.

1a. Let X be a measure space with measure μ. Let $f \in L^1(X, \mu)$. Prove that for each $\epsilon > 0$ there exists $\delta > 0$ such that if A is a measurable set with $\mu(A) < \delta$, then

$$\int_A |f| d\mu < \epsilon.$$

2a. Let P be a point of an algebraic curve C of genus g. Prove that any divisor D with $\deg D = 0$ is equivalent to a divisor of the form $E - gP$, where $E > 0$.

3a. Let f be a function that is analytic on the annulus $1 \leq |z| \leq 2$ and assume that $|f(z)|$ is constant on each circle of the boundary of the annulus. Show that f can be meromorphically continued to $\mathbb{C} - \{0\}$.

4a. Prove that the rings $\mathbb{C}[x, y]/(x^2 - y^m)$, $m = 1, 2, 3, 4$, are all non-isomorphic.

5a. Show that the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ is not isometric to any sphere $x^2 + y^2 + z^2 = r$.

6a. For each of the properties P_1 through P_4 listed below either show the existence of a CW complex X with those properties or else show that there doesn't exist such a CW complex.

- P1. The fundamental group of X is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$.
- P2. The cohomology ring $H^*(X, \mathbb{Z})$ is isomorphic to the graded ring freely generated by one element in degree 2.
- P3. The CW complex X is "finite" (i.e., is built out of a finite number of cells) and the cohomology ring of its universal covering space is not finitely generated.
- P4. The cohomology ring $H^*(X, \mathbb{Z})$ is generated by its elements of degree 1 and has nontrivial elements of degree 100.

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Wednesday October 3, 2001 (Day 2)

Each question is worth 10 points, and parts of questions are of equal weight.

1b. Prove that a general surface of degree 4 in $\mathbb{P}^3_{\mathbb{C}}$ contains no lines.

2b. Let R be a ring. We say that Fermat's last theorem is false in R if there exists $x, y, z \in R$ and $n \in \mathbb{Z}$ with $n \geq 3$ such that $x^n + y^n = z^n$ and $xyz \neq 0$. For which prime numbers p is Fermat's last theorem false in the residue class ring $\mathbb{Z}/p\mathbb{Z}$?

3b. Compute the integral

$$\int_0^\infty \frac{\cos(x)}{1+x^2}\, dx.$$

4b. Let $R = \mathbb{Z}[x]/(f)$, where $f = x^4 - x^3 + x^2 - 2x + 4$. Let $I = 3R$ be the principal ideal of R generated by 3. Find all prime ideals $\wp$ of R that contain I. (Give generators for each $\wp$.)

5b. Let $\mathfrak{S}_4$ be the symmetric group on four letters. Give the character table of $\mathfrak{S}_4$, and explain how you computed it.

6b. Let $X \subset \mathbb{R}^2$ and let $f : X \to \mathbb{R}^2$ be distance non-increasing. Show that f extends to a distance non-increasing map $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\hat{f}|_X = f$. Does your construction of $\hat{f}$ necessarily use the Axiom of Choice?

(Hint: Imagine that X consists of 3 points. How would you extend f to $X \cup \{p\}$ for any 4th point p?)

QUALIFYING EXAMINATION

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Thursday October 4, 2001 (Day 3)

Each question is worth 10 points, and parts of questions are of equal weight.

1c. Let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be the surface defined by the equation $XY - ZW = 0$. Find two skew lines on S. Prove that S is nonsingular, birationally equivalent to $\mathbb{P}^2_{\mathbb{C}}$, but not isomorphic to $\mathbb{P}^2_{\mathbb{C}}$.

2c. Let $f \in \mathbb{C}[z]$ be a degree n polynomial and for any positive real number R, let $M(R) = \max_{|z|=R} |f(z)|$. Show that if $R_2 > R_1 > 0$, then

$$\frac{M(R_2)}{R_2^n} \leq \frac{M(R_1)}{R_1^n},$$

with equality being possible only if $f(z) = Cz^n$, for some constant C.

3c. Describe, as a direct sum of cyclic groups, the cokernel of $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by left multiplication by the matrix

$$\begin{bmatrix} 3 & 5 & 21 \\ 3 & 10 & 14 \\ -24 & -65 & -126 \end{bmatrix}.$$

4c. Let X and Y be compact orientable 2-manifolds of genus g and h, respectively, and let $f : X \to Y$ be any continuous map. Assuming that the degree of f is nonzero (that is, the induced map $f^* : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is nonzero), show that $g \geq h$.

5c. Use the Rouché's theorem to show that the equation $ze^{\lambda - z} = 1$, where λ is a given real number greater than 1, has exactly one root in the disk $|z| < 1$. Show that this root is real.

6c. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function such that for all x and $y \neq 0$,

$$\frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|} \leq B,$$

for some finite constant B. Prove that for all $x \neq y$,

$$|f(x) - f(y)| \leq M \cdot |x - y| \cdot \left(1 + \log^+\left(\frac{1}{|x-y|}\right)\right),$$

where M depends on B and $\|f\|_\infty$, and $\log^+(x) = \max(0, \log x)$.