Tilemee's lecture


\[ \mathcal{H}^N = \bigotimes_{\ell \in N} \mathbb{Z} \left( \frac{T_{\ell, i}}{T_{\ell, i}} \right) i = 0, 1, 2 \]

Eigen system:
\[ \Theta_p : \mathcal{H}^N \rightarrow \mathbb{C} \text{ ring hom.} \]

If \( \Psi \) is even, \( \chi = (\chi_1, \chi_2) \), \( \chi_1 \neq \chi_2 \)

\[ \text{Im} \Theta_p \subseteq \mathcal{O}_E, \quad E \text{ number field} \]

Hecke polynomial
\[ P_{\Psi, \ell} \in \mathcal{O}_E[\mathbb{C}] \]

\[ \mathcal{H}^N[\mathbb{C}] \supseteq P_{\Psi} \]

\[ \Theta_p \downarrow \mathbb{I} \]

\[ \mathcal{O}_E[\mathbb{C}] \supseteq P_{\Psi, \ell} \]

\[ U_\ell = M(z) \left( \begin{smallmatrix} 1 & \ell \\ \ell & 1 \end{smallmatrix} \right) M(z) \in \mathbb{Q} \left[ \frac{M(z)}{\mathcal{O}_E[\mathbb{C}]} \right] \]

Hecke Functors

\[ W_0 \uparrow \text{ Satake} \]

\[ \mathcal{H}_\ell \subset \mathbb{Q} \left[ G(z) \right] \]

\[ M = \left( \begin{array}{cc} A & 0 \\ 0 & A^{-1} \end{array} \right) \in \text{Sp}_n \]

\[ P_{\ell, \ell} = \text{Tr} r(X; U_\ell, Q \left[ G(z) \backslash G(Q) / G(z) \right]) \]

\[ = X^{n-1} - T_{2, i} X^{n-2} + \ldots + t_{\ell}^{2m} T_{2, i}^2 \\ + t_{\ell}^{2m+1} T_{2, i}^{2m+1} \ldots \]
Andreoun, 3. 3. 35.

\[ P_t, e \rightarrow \text{prime-to-} N\text{-part of} \]
\[ \text{degree four automorphic} \]
\[ L\text{-function of } f \]

\[ L^{(w)}(f, s) = \prod_{p \mid N} \frac{P_t, e \left( L^{(w)}ight)}{1 - a_p L^{(w)}(f, s)} \]

Fix a prime \( p \), \( f \in S_q(N) \). \( K \) cohomological

\[ \mathbb{Q} \rightarrow \overline{\mathbb{Q}}_p \]

\[ \text{Theorem: (R. Taylor, Lazer, E. Tenenbaum, Ast 302)} \]

\[ \exists F, p\text{-adic field, } F \supset \mathbb{Q}(E) \]

\[ \exists p, q : \mathbb{Q}_p \rightarrow \mathbb{Q}/(\mathbb{Q}) \rightarrow \mathbb{G}_L(F) \]

\[ \text{semisimple, unramified outside } N_p \]

\[ \forall l \mid N_p, \text{ det} (X_{1, 1} - \rho_{l, p}(F_{\text{red}})) = P_{\text{red}}(l) \]

It is conjectural that if \( l \) is not "particular"

then:

\[ (S_{\text{sym} \lambda}) \rho_{l, p} \text{ takes values in } \mathbb{G}_{Sp_4(\mathbb{F})} \] s.t.

\[ S = \mu^0 F = X^{-\left(kl \cdot l_2 - 3\right)} \mathbb{W}^{\mu_0} \]

\[ \mathbb{W_p : (\mathbb{Q}/\mathbb{Q})^X \rightarrow \mathbb{C}^X}, \quad f| \sigma \cdot \mathbb{W}(a) \sim \]

\[ \sigma^2 \left( \frac{q}{a} \right) \left( \frac{a}{a} \right) \sim (l) \]
Aside:

Type \( p_1 \):

\[ \pi \Rightarrow \eta \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ \ast \end{pmatrix} \ (N) \]

Using \((-1)^n\):

E.g., \( \Gamma_{1, B} (N) = \begin{pmatrix} \delta \equiv \begin{pmatrix} 1 \\ \ast \\ \ast \\ \ast \end{pmatrix} \\ \ast \end{pmatrix} \) \ (Jnd \ and \ N) \]

\[ \begin{array}{c}
\Gamma_{0, P} \\
\subset \\
\Gamma_{0, B} \\
\subset \\
\Gamma_{0, Q} \\
\subset \\
\Gamma_{1, P} \\
\subset \\
\Gamma_{1, Q} \end{array} \]

\[ \begin{pmatrix} \ast \\ \ast \\ \ast \end{pmatrix} \]

\( \xi (\mathbb{Z}_E) \subset \xi' (\mathbb{Z}_E) = \Pi_{p_1} \]

\( \xi' (\mathbb{Z}_E) \)

\((\text{Inv}) \quad \Pi_{p_1} \) is abs. irreducible.

Remark: If \( \pi \) has "global mult." 1, then (Sympl) holds.

If the motivic weight of \( \pi \), \( k + k_\pi - 3 < \frac{d+1}{2} \),

then \((\text{Inv})\) holds.
Conj. If $F$ is not particular, then it has global multi. 1.

Assume from now on that $F$, $t$ is symplectic and abs. irreducible.

In particular, $\nu(p_{F,t}(c)) = -1$.

$$p_{F,t}(c) \sim \begin{pmatrix} 1 & \gamma \nu(p(c)) & \nu(p(c)) \\ \nu(p(c)) & 1 \end{pmatrix}$$

Conjecture: $F$ finite field of char. $p$.

$$\bar{\rho} : G_F \rightarrow GSp_4(K)$$

(continuous)

If $\bar{\rho}$ abs. irreducible and odd, then $\exists \sigma \in Sk(F)$, $d$-cogen.

s.t. $\bar{p}_{F,t} = \bar{\rho}$ (Has to be made more precise.)

(Reform between modularity and rank is not clear.)

Conjecture (generalized modularityconj.)

$$p : G_F \rightarrow GSp_4(F),$$

(cont. abs. irreducible, geometric)

1) If HT weight are regular, $\alpha < b < c < d$, and $t = b + c$,

then $\exists \kappa$ cohom., $\exists \psi$ eigen. in $Sk(F)$

s.t. $\rho = p_{F,t} \circ \chi^\kappa$

$$b - a = k_2 - 2$$
$$c - a = k_1 - 1$$
$$d - a = k_1 + k_2 - 3$$
2) relative singular HT weights \((a, a, b, b), a < b\)
\[ E \subset S_{z, b, a+1}(P') \quad \rho^2 \rho_{2, p} \otimes X^a \]

3) totally singular, \((a, a, a, a)\).

No holomorphic Siegel modular form should realize \(\rho\). (or no cohomologically \(SM^{12} \ldots\))

Start with \(\rho\) (in absence of Serre's conj.)
\[ \rho^2 \rightarrow GSp_4(P) \]

Assume:
\[ \overline{\rho} = \overline{\rho}_{2, p}, \quad f_0 \in \text{Sk}(P), k \circ \text{Cohom} \]

After a long list of assumptions on \(p\) and \(f_0\), one concludes \(p^2 \rho_{2, p}\)

with \(g\) s.t.

1) \(H^* HT(p)\) are regular, \(0 < b < c < d\),
\[ g \in \text{Sk}(P \cap \Gamma_2, \Gamma(p)) \]
\(k\) determined by \(b, c, d\) as above.

2) \(H^* HT(p)\) is relatively singular, \((0, 0, 1, 1)\)
\[ g \in \text{Sk}^+(P) \] (over convergent)

3) \(H^* HT(p)\) is totally singular, \(b - a > 1\),
\(g\) is generalized \(p\)-adic modular form (Katz)

(W.l.o.g. assume \(a = 0\))

Proved by Tjoumey
\[ K_0 \cong (K_{01}, K_{02}) \]
\[ p-1 > k_{01} + k_{02} - 3 \quad \text{(cf. p-1 > k-1 in the case)} \]
\[ p+1 = \text{level } (\Gamma) \]
\[ \text{p nearly ordinary } \quad \text{(for which Hida's theory works)} \]

\[ \text{Minimality of } \overline{p}_{F, \rho}, \quad \rho \quad \text{Petersson } \quad (\Gamma) \]
\[ \text{(I don't understand the detail)} \]

\[ \text{Large image of } \overline{p}^2 \quad \text{in } \text{GS}_{F, \rho} \quad \text{Sp}_4(K), \quad \text{Sp}_4(K) \text{CTing} \]

\[ \rho |_{\text{LT}} = \begin{pmatrix} \chi & \chi & \chi \\ \chi & \chi & \chi \\ \chi & \chi & \chi \end{pmatrix} \]
\[ \chi_i = X^{-\omega}, \theta_i \]
Errata, precisions and complements in lecture 2, J. Tilouine

1) The (twisted) Satake homomorphism

\[ \mathbb{Q} \left[ G(\mathbb{Z}_e) \backslash G(\mathbb{Q}_e) / G(\mathbb{Z}_e) \right] \rightarrow \mathbb{Q} \left[ M(\mathbb{Z}_e) \backslash M(\mathbb{Q}_e) / M(\mathbb{Z}_e) \right] \]

defines a non-Galois extension of degree four, generated by the "Hecke Frobenius" \( U_e = [M(\mathbb{Z}_e) \left( \frac{1}{\mathbb{Z}_e} \right) M(\mathbb{Z}_e)] \).

Then, \( P_e \) is defined as \( \text{Irr}(X; \mathbb{Q}, U_e; \mathbb{Q} \left[ G(\mathbb{Z}_e) \backslash G(\mathbb{Q}_e) / G(\mathbb{Z}_e) \right]) \).

2) After the existence theorem for the degree four Galois representation \( \rho_{f, p} : G \rightarrow \text{GL}_4(F) \) associated to an arbitrary cusp eigenform of cohomological weight, one should list the following remarks:

1) It is conjectured that this representation is always symplectic, with similitude factor \( \chi \simeq (k_1 + k_2 - 3) \text{gal} \) where \( \psi_f \) is the Dirichlet character defined as the finite part of the central character of \( f \).

2) If \( f \) is not "particular", one conjectures \( \rho_{f, p} \) absolutely irreducible.

3) "Particular means either CAP (= Saito-Kurokawa lift) in which case \( \rho_{f, p} = \psi \oplus \rho \oplus \psi' \) where \( \rho \) is a cusp eigenform in \( G(\mathbb{Z}_2) \) and \( \psi, \psi' \) are 1-dim. reps.

or \( f \) is a (weak) endoscopic lift from \( G(\mathbb{Q}) \times G(\mathbb{Q}) \) in which case \( \rho_{f, p} = \rho \oplus (\rho, \psi) \)

\( f, \rho, \psi, \rho_1, \rho_2, \psi_2, \psi' \)

where \( \psi \) is a 1-dim. rep.
let $f \twoheadrightarrow \pi^{\infty} \otimes \pi^{\text{hol}} = \pi$ cuspidal rep. of $GSp_4(\mathbb{A})$.

let $K$ be a congruence subgroup of $GSp_4(\mathbb{Z})$ corresponding to $\Gamma \subset GSp_4(\mathbb{Z})$.

Theorem: If $f$ is not "particular", $\Theta$ occurs in $H^3(Y_{\Gamma}, V_{\alpha}(\mathbb{C}))$. Moreover

if $m^{\text{hol}}(\pi^{\infty}) = \text{mult} (\pi^{\infty} \otimes \pi^{\text{hol}})$

and $m^{\text{wh}}(\pi^{\infty}) = \text{mult} (\pi^{\infty} \otimes \pi^{\text{wh}})$

then

$$4 \cdot \dim \left( \frac{H^3(Y_{\Gamma}, V_{\alpha}(\mathbb{C}) \{[\Theta_\pi] \}}{\text{K}} \right) = 2 \left( m^{\text{hol}}(\pi^{\infty}) + m^{\text{wh}}(\pi^{\infty}) \right) \cdot \dim(\pi^{\infty})^K$$

Definition: $\pi^{\infty}$ has multiplicity one if $m^{\text{hol}}(\pi^{\infty}) = 1$ or

$m^{\text{wh}}(\pi^{\infty}) = 1$

$\pi'$ is weakly equivalent to $\pi$ if for almost all prime

$\pi'_f \cong \pi_f$

Proposition: If $f$ is not particular and $\pi^{\infty}$ is weakly equivalent to a multiplicity one representation, then

$m^{\text{hol}}(\pi^{\infty}) = m^{\text{wh}}(\pi^{\infty}) = 1$

and $f_{\Theta, \pi}$ is symplectic.

Conjecture: If $f$ is not particular, $\pi^{\infty}$ is weakly equivalent to a multiplicity one representation

Theorem: If $f$ is not particular and $\pi^{\infty}$ is weakly equivalent to a multiplicity one rep., then $\rho_f$ is Hodge-Tate with weights $0, k_2 - 2, k_1 - 1, k_1 + k_2 - 3$. 