3.10. \( K/\mathbb{Q}_p \) fin.

v. spaces are always over \( K \).

(adm) smooth rep of \( G \) on \( V \). \( \text{"adm" somehow disappeared} \)

\( \forall \text{opt open subgp } U \subseteq G, \text{the fixed vectors } V^U \text{ is fin dim.} \)

Banach space reps

\( G \) is any p-adic Lie gp. (e.g. \( G = \text{GL}_n(L) \) or any open subgp therein)

**Def** A \( \text{[norm]} \) on a \( K \)-v. sp \( V \) is a fn \( \| \cdot \|: V \to \mathbb{R}_{\geq 0} \) s.t.

- (i) \( \| u + v \| \leq \max \{ \| u \|, \| v \| \} \)
- (ii) \( \| au \| = \| a \| \cdot \| u \| \)
- (iii) \( \| u \| = 0 \Rightarrow u = 0 \)

gives a metric and hence a top on \( V \).

**Def** A \( \text{[K-Banach space]} \) is a top'd \( K \)-v. sp. whose top. can be defined by a norm and which is complete.

**Def** A \( \text{[Banach rep]} \) \( V \) of \( G \) is a lin. action of \( G \) on \( V \) s.t. the map \( G \times V \to V \) is cont.

\( \text{Ban}_K^G(\mathbb{G}):= \text{category of these.} \)

This is unreasonably big catep!

E.g. \( V_0 \to V_1 \) with dense image, not surj, \( V_0 \) and \( V_1 \) are top. imed \( \frac{\text{[no closed proper } G \text{-inv. subsp]}}{\text{top. irre}} \)

**Ex** (Diarra).

\( G = \mathbb{Z}_p. \)

Take \( \bar{z} \in \mathbb{C}_p = \bar{\mathbb{Q}}_p \) s.t. \( |\bar{z}| < 1 \).

\( V_{\bar{z}} \) the smallest closed subfield of \( \mathbb{C}_p \) containing \( \bar{z} \).

\( \text{a } \mathbb{Q}_p \text{-Banach sp.} \)

let \( G \) act by \( a \cdot v := (1 + \bar{z})^a \cdot v. \)

**Fact** If \( z \) is transv. to \( 1 \in \mathbb{Q}_p \), the \( V_0 \) is fin dim and is an \( G \)-irred.
Goal: Construct a "reasonable" subcategory of "admissible" Ban. rep. which is still rich enough.

Rem 1. \( V \) is a Ban. space wi norm \( \| \cdot \| \)

\[ L := \{ v \in V : \| v \| \leq 1 \} \text{ is an } O_K \text{-submod s.t.} \]

* \( K \otimes_{O_K} L = V \) ("lattice") (but not nec. free \( O_K \)-mod...)
* \( L \) is open & bdd in \( V \).

Vice versa, let \( L \leq V \) be a bdd open lattice, then \( \| \cdot \|' := \inf \{ a \ast | \text{ is a defining norm for } V \}

2. In general, we do not find a defining norm which is preserved by \( G \). But we do for any open subgp \( H \leq G \).

Q: \( \| \cdot \|' \) is equiv to \( \| \cdot \| \), and these are discrete valn?}

### Key def
A Banach rep \( V \) of \( G \) is called \( \overline{\text{adm}} \)

if \( \exists \) a cpt open subgp \( H \leq G \) and an \( H \)-inv. bdd open lattice \( L \leq V \) s.t. the following finiteness condition holds:

\[ \forall \text{open subgp } U \leq H \text{ the } U \text{-fixed elts } (V/L)^U \]

are of cofinite type over \( O_K \), "obj. obj".

Rem
1. "cofin type" means \( (K/O_K)m \) is finite. (Post dual is)
2. If \( V \) is adm, then \( \forall \) cpt open subgp \( H \leq G \), we find...
3. Suppose \( V \) is adm, \( \forall H \) and \( L \) as in the def.

reduce to \( \Xi = L/\Pi_K L \) is a smooth \( H \)-rep over \( O_K/\Pi_K \)

which is adm-smooth, (can go backward from \( \Xi \) to \( L \)) by Nakayama - Mazur.

\[ \text{Ban}_k (G) := \text{categ of all adm ones} \]
Thm (Sch./Teitelbaum)
(i) $\text{Ban}_K^a(G)$ is an abel. categ.
(ii) All maps in $\text{Ban}_K^a(G)$ are strict w. closed image.
\[
\circ: V_1 \to V_2 \to V_3 \to 0 \Rightarrow \text{top. on } V_3 = \text{top. on } V_2 / V_1
\]
(Strategy of pf)
May assume that $G$ is opt.

$C(G) := \{ \text{all cont. fun } G \to K \}$

$\tilde{g}: \text{Ban. rep for sup-norm.}$

$g \mapsto \tilde{g}_y \in D^c(G, K) := \text{cont. dual of } C(G).$

Fad 1 (Lazard) - serious thm.

$\mathcal{O}_K[G]$ is noeth.

Fact 2 (main step).

$\text{Ban}_K^a(G) \xrightarrow{\sim} \text{Mod}_{\mathcal{O}_K[G]}(D^c(G, K))$

$V \mapsto V' = \text{cont. dual of } V.$

is an anti-eqv. of categ.

Fact 2 (main step).

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cont. principal series

$G = G_{L_n(Q_p)}$

$U_1$

$P := \text{lower } \triangle \text{ matrices.}$

$\chi: P \to K^\times$ cont. char.

$\text{Ind}_P^G(\chi) = \{ f: G \to K \text{ cont } \mid f(gp) = \chi(p)^4 f(g), \forall g \in G, p \in P \}$

$G$ acts by left transl.

why Ban. rep?) Iwasawa dec. $G = G_0 P, \ G_0 := G_{L_n(Q_p)}$

take sup-norm/conv $G_0 \to \text{Ind}_P^G(\chi)$ is an adm. Ban. rep.
\[
\text{why adm?}
\]
\[
\text{Ind}_{p^2}(X) = \text{Ind}_{pG_0}(X) \quad \text{dual.}
\]
\[
\downarrow
\]
\[
\text{C}(G_0), \quad \text{D}_{c}(G_0, k).
\]

\[n=2\]
\[
X(A, a) = \exp(c(x)/\log(a)) \text{ for } a \text{ close to } 1.
\]

where \(c(x) \in k\).

**Thm** If \(c(x) \neq -1/\alpha_0\), then \(\text{Ind}_{p^2}(X)\) is top. irreducible for \(G_0\) (Sch.Tate-Teitel) in gen'l, for \(G\) the length is \(\leq 2\).

3.15. \[
X(\mathcal{O}_k) = \text{str. sst.}
\]
\[k = \mathcal{O}_k/(\alpha_i).\]
\[
\mathcal{O}_{\text{UR}} \quad \text{rel dim } n.
\]
\[
Y = X \otimes \mathcal{O}_k k = \bigcup_{i=1}^{n+1} Y_i.
\]

proper cmk / k.

irred. dim n.

\[I \subset \{1, 2, \ldots, n+1\} \]
\[
Y_I := \cap_{i \in I} Y_i.
\]

\[
\text{rel dim } \frac{Y_I \otimes \mathcal{O}_k}{\mathcal{O}_k} \quad \text{dim } 2n.
\]

and not regular.

not normal.

crossing.

\[
X', \quad \text{str. sst.}
\]
\[
\downarrow \quad \text{Saito's blow-up}
\]
\[
X' \otimes \mathcal{O}_k k = \bigcup_{1 \leq i, j \leq n+1} \text{Cart. div. crossing normally with each other.}
\]

\[
\text{Str. transf. of } Y_{ij} \text{ in } X'
\]

\[
\text{Irred. comp. of } Y' = X' \otimes \mathcal{O}_k k.
\]