Eigenvarieties at some non-Hopf points and Selmer groups (with J. Bellaïche)

1) A conjecture of Bloch-Kato

$E/A$, $g$ imaginary, $\pi$ cuspidal and rep of $\text{GL}_n(A_E)$

i) $\pi^0 \cong \pi^c \oplus 1$

st.

ii) $\pi$ no algebraic regular ($\kappa_{\text{al}} \not\in \frac{1}{2}$)

Assume $f : G_E \to \text{GL}_n(\overline{\mathbb{Q}})$ compact LLC outside $p$, pure...

\[ \Rightarrow \quad L(\pi, 0) = L(\pi, 1) \]

Conj $10^1$. and $L(\pi, 1) = \det H^1_f(E, \rho)$

(weaker), $L(\pi, 0) = -1 \Rightarrow H^1_f(E, \rho) \neq 0$

Assumptions

i) $p = w$ splits in $E$, $\pi$ unramified.

ii) $\pi_w$ ramified $\Rightarrow w$ splits in $E$ (not poss $p$-old, forget it to simplify)

iii) $4/n$ and $0, -1$ are not HTN of $S$ if $D_v$

Theorem 1: $\text{Rep}(n+2) : AC(\pi) \Rightarrow$ sign conj.

Theorem 2: Assume $V + \{ i \}$ $H^1_f(E, \rho(-i)) = H^1_f(E, \text{ad } \rho) \neq 0$

i) $\lambda \rho$ and all $\text{isc} = n$

iii) $\pi_v$ has a reg. refinement, non-critical

Then for some explicit $E_{3, x}$, $\dim \overline{\mathcal{L}}_x(E) \leq n \left( m + \frac{n+1}{2} \right) + 1$

eigen of $U(n+2)$

$\det \psi \subseteq H^1_f(E, \rho)$

Thus we can conclude from $(E, x)$

Coroll: $\pi = 1 \Rightarrow E$ smooth at $x$. 
Remarks: Sign conjectured known \( n = 1 \), \( \text{Hecke} \), \((BC, \Lambda, \text{Adele}, \text{EV} 2001)\) \\
\( n = 2 \) coming from mod forms, Hecke's, \( \text{Sk. Adele, same case.} \) \\
parity \\
- concentrate on \( \text{Kum}_2 \), nothing is conjectural \( n = 1 \) \\
- good choices of \( E \), start from mod forms \( \text{Rep} (4) \) \\
- is main, missing

2) The conjecture \( \text{AC}(\pi) \)

\( \mathbb{Q} \times \mathbb{Q} \rightarrow \exists \ U(n+2) / \mathbb{Q} \) q.spl. all finite places, rich points compact \( \text{Hans } p \).

Endoscopic Functoriality \( L \ U(2) \times \mathbb{L} U(n) \rightarrow L \ U(n+2) \)

Arthur conjectures that \( \Pi \) should transform to an "A-packet" of \( \text{Rep} \ (\Pi) = \frac{1}{2} \pi(y) \) of \( U(n+2) \), whose Galois rep. (assuming \( \text{Rep}(n+2) \) is \( 1 + \omega + p \) (not tempered) has \\
\( * \rightarrow \text{not tempered, dis} \).

- Interested in a special element "base element"

Let \( \pi_0 = \pi_{\omega} \otimes \pi_v \) the obvious rep attached to \( 1 + \omega + p \) by LLC. \\
\( \text{cong} \ (\text{AC}(\pi_1)) \quad \pi_0 \leftrightarrow L^2 (U(n+2)(A) \backslash U(n+2)(A), \mathbb{C}) \text{ if } E(\Pi,v) = -1 \)

- \( \text{Actually iff, and if it is the case, multiplicity should be } 1 \}
- \( \text{Know } n = 1 \), other cases? \( \text{(Kagawa?)} \)
- \( \text{In Bell's lecture, he will maybe describe } \Pi \text{ when } n = 1. \ (\text{Kagawa?}) \)
Rough idea where "the Selmer ele"s come from"
along the lines of Ribet's work on converse of Herbrand's thm.
idea to use these \( T \) when \( E(\Gamma_0) = -1 \) due to several people (Hans). 

Bellaïche Thm (2001) \( m = 1 \).

i) Deform \( 1 + w + f \) to some \( \rho' \) att. to a stable rep. \( (\rho') \) form on \( U(3) \).
   (congruence, level raising)

ii) Lattice argument, \( 3 \) fadles, at least a serious issue that
   \( R \) doesn't have
   to produce \( 1 \) by \( f' \), we must show that \( 1 \) by \( w \) (e.g.) does not appear.
   show it is "f" and use that \( E^{\times} \) finite. (Kunnik).

Shimura-Uchida \( \sim \) GSp_4 use families to produce deformation alg...so in \( \text{Wls} \) also.

iii) discover cannot use red. family, \( \pi \) not required.

4) Proof Thm 2

\[ E(\Gamma_0) = -1, \quad A(C) \Rightarrow \exists \pi_0 \rightarrow L^2 \left( U(n_1)(\mathbb{Q}), \frac{U(n_2)(\mathbb{A})}{U(n_2)(\mathbb{Q})}, \mathcal{C} \right) \]

choose a minimal level \( \Gamma_0 \) for \( \pi_0 \) (type \( B_k \))

[\[ H = \mathbb{A} \otimes \text{spherical at all } \mathfrak{p} \text{ away from places } \mathfrak{p} \]
\[ \rightarrow \text{ eigenvar. } E, \quad \text{Rep}(n_2) \rightarrow \mathcal{T} : G_E \rightarrow \mathcal{O}(E) \]

choose a point \( \pi \) choose a \( \rho \)-refinement of \( \Gamma_0, V \)

\[ (A, \rho^0, \rho_1, \ldots, \rho_n, \rho^1) \]
\[ \{ \text{accountable as } 1 \text{ proceeds } \rho^1 \}
\[ \text{rank of } \rho \]
\[ T : G_e \to A, \quad A = \mathcal{O}_x, \quad m = m_{x_1}, \quad K = \text{Frac} A\]

\[ T \mod m = 1 + \omega + \varphi \]

analyse the galois rep. quite carefully, in the span of \( MW, W \).

we want to compute \( S := \frac{A[G]}{K \pi T} \)

\[ S = \left( \begin{array}{ccc}
M_0(A) & A_{wp} & A_n \\
A_{p_0} & A & A_{p_1} \\
A_{p_1} & A_{wp} & A
\end{array} \right) \subset \mathcal{M}_{2n+1} \quad (K1) \]

Lecture 3

Moreover \( \tau : g \mapsto c(g) \omega(\tau) \)

factor through \( \tau : S \to S \) anti-involution.

\[ \tau f = f^{-1} \quad \tau \circ \omega \circ \tau = \omega \circ \tau \circ \omega \]

Lemma about ideles (in the lifting). \( \tau \) induces map \( A_{i,0} \cong A_{u_0,\mathfrak{c}(i)} \)

Aim: compute \( A_{i,0} \).

(4) Reduced roots

Lemma: the total red. root of \( T \) is \( m \).

Use Lecture 6, reducible point \( T \mod m = 1 + \omega + \varphi \)

in this order, refinement is integral. \( \{1, \varphi, \omega, \varphi \} \)


compute permutation \( \sigma \) weights \( k_1 < k_2 < \cdots < 1 < 0 < k_{i_2} \cdots k_{m2} \)

\[ i \rightarrow i, 1 \]
\[ 2 \rightarrow i \]
\[ 3 \rightarrow 2 \]
\[ \varepsilon \rightarrow i \]
\[ i+1 \rightarrow i+2 \]
\[ i+2 \rightarrow i+1 \]

It is easy.
Hence by Sec. 6. \( A_{\mathfrak{m}} \) is finite length

\[ T \mod m = \sum \tilde{\psi}_{\mathfrak{m}}^{\ast} + \mathfrak{m} \tilde{\psi}_{\mathfrak{m}}^{\ast} \]

\( \Lambda \rightarrow \) 

all crystalline deformations.

\[ \text{but} \quad H^1_\ast(E, \mathrm{ad} \phi) = 0 = H^1_\ast(E, \phi) \quad \text{all are} \]

\[ \text{finite} \quad \text{ideal of class group}. \]

ii) whether are ok if one is.

\[ \Theta_{\mathfrak{m}, x} \] is generated over \( \Theta_{\mathfrak{m}, x, \mathrm{cm}} \) by \( \mathfrak{m}'s \) \( \mathfrak{u}_\mathfrak{p} 's \Rightarrow \mathfrak{I} = m \Rightarrow \mathfrak{I}_{\mathfrak{m}} = m \]

Corollary

\[ m = A_{\mathfrak{m}} A_{\mathfrak{m}} + A_{\mathfrak{p}} A_{\mathfrak{p}} + A_{\mathfrak{p}} A_{\mathfrak{p}}. \]

B) Ext comp

\[ \text{Ext} \quad \text{comp} \]

\[ \text{Ext}_{\text{Sec}}(P, \delta) \xrightarrow{\sim} \text{Hom} \left( \Lambda_{\mathfrak{m}} / \Lambda_{\mathfrak{m}} \mathrm{ad} \phi, A_{\mathfrak{m}} \right) \]

\[ \text{Ext}_{\text{Sec}}(P, \delta) \]

\[ \text{Ext}_{\text{Sec}}(P, \delta) \]

\[ \text{(iii)} \text{ box about Ext}_{\text{Sec}}(P, \delta) \]

1) All tresses Ext fall into \( \delta \)-cabled pair

ii) at \( \mathfrak{p} \)

ii) duality.

iv) dual are bounded by

1) Outside \( \mathfrak{p} \), BK lies at some auxiliary

ii) \( \mathfrak{A} \quad \mathfrak{p} \quad \text{2 places} \quad v \rightarrow \mathfrak{A} \quad \mathfrak{p} \)

+ duality

iii) properties of \( T \).

abstract
\[ H^1_\ell(E, Q_p(1)) = Q^\times \otimes Q_p = 0 \]
\[ H^1_\ell(E, Q_p(1)) = 0 \quad \text{(scale, MW)} \Rightarrow \text{ass. 1 by } w' \]
\[ H^1_\ell(E, p(1)) = 0 \quad \text{by assumption} \]

local comp. using fundam. exact sequence \[ \mathbf{0} \xrightarrow{\varphi_{\ell,1}} \mathbf{0} \xrightarrow{\mathbf{1}} \mathbf{0} \]
\[ \Rightarrow \text{dim } H^1_\ell(E, p(1)) \leq n. \]

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3) End argument

\[ \text{NAK} \Rightarrow A_{\omega} = A_{\ell, p} A_{p} \omega \quad A_{\omega} A_{\omega} \subset A_{\ell, p} A_{p} \]

\[ \text{duality} \Rightarrow A_{p} \omega A_{\omega} = A_{\ell, p} A_{p} \]

\[ \Rightarrow \text{dim } = m = A_{\ell, p} A_{p} \]

\[ A_{\omega} A_{\ell, p} = A_{\ell, p} A_{p} \omega \quad \text{m} \quad A_{p} = \text{NAK} \]

\[ \text{min} \quad A_{\ell, p} = \text{dim } \mathbf{E}_\text{rat}(1, p) = \text{max number of indep. ext. of } 1 \text{ by } p \quad \text{we can find in lattice of } \mathbf{K} \text{.} \]

\[ A_{p, 1} = \sum_{i=1}^{m} A_{p, i} + A_{p} \omega A_{\omega} \]
\[ \text{and } A_{p} \omega A_{\omega} = \prod_{p} A_{p} + A_{p} \omega A_{p} \]

\[ \Rightarrow A_{p} \omega A_{\omega} = \prod_{p} A_{p} + A_{p} \omega A_{p} A_{\omega} A_{\omega} A_{p} \]

\[ \Rightarrow \text{m } h_{0} < (n + n) \]

\[ m = A_{p} \omega (\sum_{i=1}^{m} A_{p, i}) + \prod_{p} A_{p} \text{ } A_{p} \]

\[ \Rightarrow \text{dim } \mathbf{H}_0^{1}(\mathbf{E}, q(1, 1, 1)) \]
Prakto

* find examples to know! (computable)
* link with $p$-adic $L$-functions
* does not follow from Mani Cony.