Triangular properties of the family of Galois rep. on the eigenvarieties

I) Setting.

Fix $G$, definite unitary gp. attached to $E/\mathbb{Q}$, im field, rank $d$.
$p = v_0$ split prime, $G(\mathbb{Q}_p) \cong GL_d(\mathbb{Q}_p)$.

$K = \prod K_v$ compact open $\subset G(\mathbb{A}^\infty)^e$, $K_p = 1$ when $v_p \not\in S$, maximal henselian.
$e \not\in S$ finite.

$H = \mathfrak{A} \otimes \mathbb{Z} \begin{bmatrix} K^\times \setminus G(\mathbb{A}^\infty) & K^\times \setminus K^\times \end{bmatrix}$

$I = S^{-1}J$

Recall

rigid space $\mathfrak{A}^S$

$\Psi : H \to O(E)^{\leq 1}$ ring hom.

$Z \subset E$ Zariski dense subset

such that:

- $k(\eta) = (k_1(\eta) \leq \ldots \leq k_d(\eta))$
- $z \in Z \Rightarrow \Psi_z$ is the system of eigenvalues of $H$

on an eigenform $f \in G, \text{ level } K$.

$W \times \mathbb{G}_m$ $p$-refinement of the ad. rep. generated by $f_z$.

weight

$(k_1(\eta), \ldots, k_d(\eta))$

$(a, b) \mapsto \begin{bmatrix} k_1(a) & k_2(a) \end{bmatrix} x^T$

$k_1(\eta) \leq \ldots \leq k_d(\eta)$

are the roots of the Hecke pol. accessible

$\Psi_1(\eta), \ldots, \Psi_d(\eta)$
\( z \in \mathbb{Z}, \exists \text{ so galois rep. } \beta_i : G_{E,S} \to GL_d(\overline{\mathbb{Q}}_p) \)

\[ T_i \beta_i(\mathbb{F}_v) = T_i(z) \neq 0 \text{ for } v \neq S. \]

\[ S_i = G_{E_i} = G_{E_0} \text{ in crystalline, HT weights } k_i(z) < k_i(21) < \ldots < k_i(n) + 1 \]

(geometric conventions)

\[ E_i, \beta_i : \mathbb{F}_i(z) = \mathbb{F}_i^{\text{HTW}} \]

NB: Change a bit \( K_i \) such that they give exactly the HTW.

i.e. \( k_i(z) \) are HTW of \( S_i \).

At such \( z \), assume we have \( \beta_i(z) \) are 2 by 2 matrices.

As refinement of Dary (\( S_i \mid G_{E_0} \)), we will call it \( F_i \).

\( T_i \) : Global Galo pseudo-cancel by restriction at \( p \rightarrow T_i G_{E_0} : G_{E,S} \to \mathbb{Q}(E). \)

Refined family of Galo's representations. (???)

\section{II Regular crystalline classical points}

Fix \( z \in \mathbb{Z} \), study of the family around \( z \), \( A = \mathbb{Q}_{E,z}^{\text{reg}} \)

\section{Assume (REG)} \( F_i \) is non critical, regular (local)

\( \forall i, \beta_i(\mathbb{F}_i) \) is a simple eigenvalue of \( \text{End} \) on \( \Lambda^i F_i \).

(IRR) \( \Lambda^i F_i \) is indep. \( \forall i \leq d \). (global).

In particular \( \exists \beta : G_{E,S} \to M_d(A) \text{ trace } T. \)

(strictly speaking, as alg., we call it by act. no field, not important.)

\( \text{Def} \) \( S_i : \mathbb{Q}_p^x \to \mathbb{A}^1 \), \( S_i(p) = F_i \)

\( S_i |_T = k_i^{-1} \).

\textbf{Thm:} For each \( i \) \( \mathcal{I} \) a cofinite length ideal, \( \mathcal{P} \otimes \mathcal{I} \) is a triangular def. of \( (\beta_{\mathcal{P}, E_i}, F_i) \) whose parameter is \( (S_i)_{\mathcal{I}}. \)
6. $f$ extends to a neigh. $\to GL_2(\mathcal{O}(W))$

5. $\phi^T = \phi \otimes K_1$ has a HT W O $F_i : \mathbb{A} \to \mathbb{A}^1$ an analytically non-
eigenvalue of $\text{Lie}(\text{flat} f) \simeq \mathbb{A}$

4. Applying Kain's contraction $\Rightarrow \sqrt{\text{Div}(\phi)} \phi = F_i$ is generically ch.\n
Enough to get $\text{Div}(\phi^T) = 0 \forall y \in Y$ but not to get that.\n
\[ \text{Div}(\phi^T) \text{ is free he 1 over } \mathbb{A} \]

In fact, yes to do this blow up the problematic ideal and descend the crystalline
period. after. \[ \hat{\mathcal{O}} \]

Use that $\text{Div}(\phi^T) = F_i(\mathbb{A})$ for this.

3. Apply some construction to $\mathcal{i} \cong \{0, \ldots, 1\} \forall i \in I$, and use.

Prop. Let $(V, F)$ be a non-critically defined $V$ with $k_1, \ldots, k_d$, $\psi_1, \ldots, \psi_t \in \mathbb{A}^1$

Let $V_{i_1}$ be a crystalline rep. (over $\mathbb{A}$ say) equipped with a refinement $\tilde{F}$
non-critical and regular. Let $V_i$ be a deformation of $V$ and assume
there are cont. $\Phi_i : \mathcal{O}_{\hat{\mathcal{O}}} \to A$ such that $V_i$

1. $\text{Div}_i \mathcal{O}_{\hat{\mathcal{O}}} \psi = F_i(\mathbb{A}) \text{ is free he 1 over } \mathbb{A}$

2. $\mathcal{O}_{\hat{\mathcal{O}}} \text{ mod m} = \mathcal{O}_{\hat{\mathcal{O}}} \psi_{i_1} \ldots \psi_t$ and $\psi = \psi_{i_1} \ldots \psi_t \psi_i$

Then $V_i$ is a trianguline def. $(V, F)$ whose params. is

$(\mathcal{O}_{\hat{\mathcal{O}}} \psi_{i_1} \ldots \psi_t)_{i=1}^t$.

(When $k_1, \ldots, k_d$ are the HT W of $V$)

Rk: Uniq this $R = \prod \mathbb{A} \{k_1, \ldots, k_d\}$ and deny

$H^1(\text{ad} \mathcal{O}_1) = 0$
Reducible points

Let \( z \in \mathbb{Z} \), anyone, \( p_z = \bar{P}_1 \oplus \cdots \oplus \bar{P}_n \) (MF as HTW are defined).

\[ \mathcal{F}_c = (\bar{P}_c(z_1), \ldots, \bar{P}_c(z_r)) \] induced refinements \( \mathcal{F}_{c_i} \) of \( \mathcal{F}_c \).

Assume that \( \mathcal{F}_{c_i} \) are intervals, and \( \mathcal{F} \) ordered, such that:

\[ \mathcal{F}_{c_{i-1}} \subset \mathcal{F}_{c_i} \subset \mathcal{F}_{c_{i+1}} \]

(Not always possible)

Define \( \sigma \in \mathcal{Q} \) associated to this combinatorial datum.

HTW are \( \bar{P}_c(z_1) < \cdots < \bar{P}_c(z_r) \).

\( k_{G(a, i, +)} \) is the smallest HTW of \( \bar{P}_c \).

\[ k_{G(a, i, -)} \]

Define: \( \bar{f}_c \in \mathcal{Q} \) associated to this combinatorial datum.

Example: \( \bar{f}_c \) have div 1, hence only characters, \( \bar{z} = 0 \).

\( k_{G(a, i, +)} \) is the HTW of \( \bar{f}_c \), i.e. \( v^+(P_c(z_i)) \).

\( 6 = i(\bar{f}_c) \) iff \( \bar{f}_c \) is the ordinary of \( \bar{f}_c \).

We 6fractions we say that 6 is anti-ordinary.

Assume

\[ (\text{Reg.) } \mathcal{F}_{c_i} \text{ non red, regular } \]

\[ (\mathcal{F}_c) \text{ mod assum. } \]

\[ \prod_{i=1}^{r} \bar{P}_i \]

\[ \forall (a_i), \; a_i \leq d. \]

\[ T \text{ mod } \mathcal{I}_{k_{c_i}} = \bar{f}_c \bar{P}_1 + \cdots + \bar{f}_c \bar{P}_n, \; \bar{f}_c : \Gamma_{E, s} \to GL_d(\Lambda/\mathcal{I}_{k_{c_i}}) \]

Theorem

Let \( \mathcal{I}_{k_{c_i}} \subset \mathcal{I}_{k_{c_i+1}} \). \( \forall i \), \( \bar{f}_c \circ \bar{f}_c \) is a trianguline deformation of \( (\bar{P}_c, \mathcal{F}_{c_i}) \) with explicit parameters, e.g.

\[ \bar{f}_c \circ \bar{f}_c \]

Moreover \( \forall a \in \mathbb{R}^{1,13}, \; k_{G(a)} \text{ is constant.} \)
Consider: Assume that \( \mathcal{E} \) is a birational, and that \( \mathcal{M} = \mathcal{M}_1 \) fulfills:

Then each \( k_i \) is of \( \mathcal{A}/I_k \), and \( \mathcal{A}/I_k \) has dual \( 0 \).

i) If \( \mathcal{E} \), \( \hom(\mathcal{F}_i, \mathcal{F}_{-1}) = 0 \), then each \( g_i \) is crystalline.

Moreover:

(ii) follows from theorem and construction of \( E \).

(iii) This follows from lemma 2.

Theorem: Basically similar to the fixed case, but extra difficulties:

coming from the fact that there is no free module \( \mathcal{O} \) over \( T \).

Use lemma 6 to get a good module \( \mathcal{M}_1 \)

structural \( \Rightarrow \) become free after a blow-up \( \overline{\mathcal{O}} \Rightarrow \mathcal{O} \) free.

get a family of const. periods above

we prove a lemma showing that \( \ell(\text{Dwyg}(\mathcal{Y})\mathcal{Y} = \mathcal{F}_1) \) is as we expect \( (\ell(\mathcal{A}/I_k)) \), and some part of \( \mathcal{M}_1 \) in free enough

to get the result.

Step (constant weight lemma)

\( \mathcal{V} \) rep of \( \mathcal{G} \), smaller in \( \mathcal{D} \).

Assume that \( \text{Dwyg}(\mathcal{V}_\lambda) \) has dual \( 1 \) and it \( \mathcal{F}_1 \) is 0.

Assume that \( \mathcal{V}_\lambda \) def. such that \( \text{Dwyg}(\mathcal{V}_\lambda)^\mathcal{Y} = \mathcal{F}_1 \), \( \lambda \in \mathcal{X} \).

over \( \lambda \) then \( \mathcal{V}_\lambda \) is a constant \( \mathcal{H} \mathcal{W} \) of \( \mathcal{V}_\lambda \).