April 13, 2006. Thursday 1:00pm. Kevin Buzzard.

16th lecture

smooth irred.

$O$-rep'n

of $GL_2(O)$

$V$

1-dim

$\leftrightarrow$

T-semi-simple

$(g, N)$ 2-dim. WP rep'n

$\leftrightarrow$

$g$ st a non-zero

$V$ exact, but $n = 0$
Local - Global compatibility

\[ f: \text{cusp form} \rightarrow \pi_g \quad \text{loc. LAN} \quad R^F \]

1-dim case

doesn't occur

At the end of last lecture:
We were considering the mod p picture:

How the non-generic case can occur in global setting?

Q) Is there a mod p local Langlands for $\text{GL}_2(\mathbb{Q}_p)$?

Observation: there is not such a correspondence, if we further demand:

1. Irreducible mod p rep'n of $\text{GL}_2(\mathbb{Q}_p) \rightarrow \text{Galois rep'n}$
2. Compatibility with reduction mod p of classical local Langlands

Dumb reason:

Existence of rep'n of $\text{GL}_2(\mathbb{Q}_p) \rightarrow \text{Aut}(V_{\mathbb{Q}_p})$

where reductions are reducible

e.g. $I(x_1, x_2)$

\[ x_1, x_2 : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times \]

\[ x_i(p) = p \text{-adic unit} \quad x_1/x_2 \equiv 1 \ (\text{mod } p) \]

Jettison: 1 reducible

Allow possibility of it's associated to $p$'s
Local Langlands is becoming a "recipe" from $\rho$'s to $\Pi$'s.

Example: if $\rho$ is cyclo $\otimes \chi$, the mod $p$ Galois rep'n $\rho_{\chi}$, then the associated $\Pi$ will have $\geq 2$ $I$-H factors.

1. $\not\equiv -1 (p)$ then 2
2. $\equiv -1 (p)$ get 3

All but 1 are $1$-dimensional and one is $\infty$-dimensional.

$I(\chi_{1}, \chi_{2}) \not\equiv I(\chi_{2}, \chi_{1})$

up mod $p$ characters of $\rho_{\chi}$ of $\rho_{\chi}$

If $\chi_{1}/\chi_{2} \equiv 1 \pm 1$

Then one has a $1$-dimensional, one had a $1$-dimensional.

Next step:

Local-Global

For overconvergent finite slope cuspidal eigenforms $\tilde{\Pi}_{f}$, p-adic HMs, rep'n of $\rho_{\chi}$

Alex Paulin has constructed $\tilde{\Pi}_{f}$ p-adic overconvergent eigenforms which are smooth admissible rep of $\rho_{\chi}$.

$\rho_{\chi}$ exists & we can ask how to relate $\tilde{\Pi}_{f} | D_{f}$ & $\tilde{\Pi}_{f}$. It seems that $\tilde{\Pi}_{f}$ is not always irreducible

Similarly $\rho_{\chi}$ irreducible. Hence no reason to be genuine.

$\Pi_{f}$ could be unram $\otimes \epsilon$.

& e.val of $\epsilon_{f}(\text{Tr}_{\overline{\mathbb{F}}}^\mathbb{Q}_{p})$ could be $\frac{\beta}{\alpha} e_{\chi} \overline{\chi}$, $\alpha/\beta = 1 - \ell$

$\Pi_{f} \otimes \epsilon$ = reducible unram $\rho_{\chi}$

C-ind $\rho_{\chi} \otimes \epsilon_{\chi}$ Stemberg in the gap.
Again it looks like there's some kind of "correspondence" $S \rightarrow T$

$T$ may not be reducible.

**Remark:** The mod $p$ story is connected to level raising & lowering.

1. If $T_l$ is a char 0 mod. form of level $N$, let $N$.

   & if $P_l$ is the mod $p$ repln

   & $P_l$ (mod 2) have evens $\alpha, \beta$, $d/\beta = l$

   then? $G_l$ level $N_l$ new at $l$.

2. If $G_l$ is a form of level $N_l$.

   new at $l$

   & if $P_l$ is unramified @ $l$ then?

   $\exists T_l$ level $N$ s.t. $\overline{P_l} = \overline{P_l}$?

yes in many cases (Mazur, Ribet)

In the $p$-adic theory,

there are analogous questions.

1. If $T_l$ is a family of Eigenforms of level $N$.

   & $l \neq N_p$ & if $P_l$ (mod 2) then evens $\alpha, \beta$

   then? $\exists G$: family of forms of level $N_l$.

   generically new at $l$.

   but s.t. $\overline{P_l} = \overline{P_l}$?

$\square = p$

Terrifying things afoot.

Let $E/Q$ be an elliptic curve with multiplicative reduc'n @ $p$.

Let $f$ be the associated modular form

$? \leftarrow \square (\text{ord} = 1)$

Q. Relate $T_{f, p}$ to $P_l | D_p$?

**Rule:** $P_l | D_p$ has determinant the cyclic char. class which is infinitely wildly ramified.
\( \Pi_{p,p} \) will be a twist of \( \Pi_{st} \). It will be unramified twist by a character of order 2.

\( \Pi_{p,p} = \Pi_{st} \) unram. good twist of \( \Pi_{st} \) (non-split)

Split multi-case:

\( \mathbb{P}_p \mid \mathbb{D}_p \) we can write it down

Tate Curve: \( E(\mathbb{Q}) \cong \mathbb{Q}_p^\times / \langle \mathfrak{p} \rangle \), \( \mathfrak{p} \in \mathbb{Q}_p^\times \), \( |\mathfrak{p}| < 1 \).

\( \mathfrak{p} \) is determined by the fact that \( j \)-invariant of \( E \):

\[ j = \frac{81}{744 + 18 \mathfrak{p} + \mathfrak{p}^2} \]

\( \mathfrak{p} \) is a global object.

\( \mathfrak{p}_f \mid \mathcal{D}_p = \begin{pmatrix} \text{cyclo} & \ast \\ 0 & 1 \end{pmatrix} \)

where \( \ast \) is an extn of \( 1 \) by cyclo determined by \( \mathfrak{p} \).

Kummer Theory

\[ \mathfrak{p} \in \mathbb{Q}_p^\times / \mathbb{Q}_p^\times \cong \mathbb{Z}_p \times \mathbb{Z}_p \]

Lots of different \( \ast \)s can occur.

Split multi-case

Rep by side \(" \Pi_{st} \"

Colors by side: only many possibility: \[ \text{Mazur-Tate-Tatelebanum} \]

Greenberg - Stevenson

The hope that we can fix things up is chalked if we move to weight 4.

Set \( p = 5 \), \( N = 45 \), \( k = 4 \): compute the new \( \Pi_{st} \) for.

One example:

\[ a_5^2 = \mathfrak{p}^{k-2} \]

\[ \mathfrak{p}_1 = \mathfrak{p}^2 - 5 \mathfrak{p}^2 + 17 \mathfrak{p} + 5 \mathfrak{p}^5 - 30 \mathfrak{p}^7 + \ldots \]

\[ \mathfrak{p}_2 = \mathfrak{p}_1^2 = \mathfrak{p}^2 - 3 \mathfrak{p}^2 + \mathfrak{p}^5 + 5 \mathfrak{p}^5 + 20 \mathfrak{p}^7 + \ldots \]

\( \Pi_{s,5} \cong \Pi_{s,5} \) unramified twist of Greenberg.
However,

\[ f_1 = \frac{\text{form of } \omega + 4 \text{ & level } \delta}{\text{mod } 5} \frac{\text{rep.}}{\text{mod } 7} \sim -8^2 + 20 \times 7 + \ldots \]

Hence \( \overline{\mathcal{P}}_{\mathcal{S}_1} \mid \mathcal{D}_3 \) is \underline{irreducible}.

\[ \text{Ind}(\omega) \]

\( \mathcal{P}_{\mathcal{S}_1} \cong \text{cyclo} \otimes \mathcal{P}_\mathcal{S}_1 \) \( \sigma \text{ at } 6 \), level 7 \( \mathcal{P}_{\mathcal{S}_2} \cong -8^2 + 6 \times 7^2 + \mathcal{P}_\mathcal{S}_2 \)

\( \mathcal{P}_{\mathcal{S}_2} \mid \mathcal{D}_3 \) is \underline{reducible} \( \quad \quad \quad \quad \quad \xi \)

\( \mathcal{P}_{\mathcal{S}_1} \mid \mathcal{D}_3 \) \( \mathcal{P}_{\mathcal{S}_2} \mid \mathcal{D}_3 \) are completely \underline{different}.

On the other hand,

T. Saito proved \underline{local-global compatibility at } \( p \) for \underline{classical modular forms}.

\[ \text{[Ind}_{\mathcal{P}}, \text{P doesn't determine } \mathcal{P}_\mathcal{S} \mid \mathcal{D}_p \text{, but } \mathcal{P}_{\mathcal{S}_1} \mid \mathcal{D}_p \text{ does determine } \text{Ind}_{\mathcal{P}}, \text{P}] \]

Primes

Let \( \mathfrak{p} : \text{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p) \rightarrow \text{Gal}(\overline{\mathbb{Q}}_p) \) be a \underline{concrete} Galois rep'n

Fontaine defined a \underline{functor} \( \mathcal{D}_{\mathfrak{p}} \)

taking \( \mathfrak{p} \) to a "linear algebra object".

i.e. \underline{filtered} \( (\mathfrak{p}, \mathcal{N}) \)-\underline{module}.

"Filtered \( (\mathfrak{p}, \mathcal{N}) \)-\underline{module}" \( D_{\mathfrak{p}}(\mathcal{N}) = (B_{\mathfrak{p}} \otimes \mathcal{V})^\mathcal{N}_\mathfrak{p} \)

1. \( \text{fin dim. } \overline{\mathbb{Q}}_p - \text{v. ap } D \)
2. A \underline{bijective} \( \overline{\mathbb{Q}}_p \)-\underline{linear map} \( \mathfrak{p} : D \rightarrow D \)
3. A \( \overline{\mathbb{Q}}_p \)-\underline{linear endo} \( \mathcal{N} : D \rightarrow D \) s.t. \( \mathcal{N} \mathfrak{p} = \mathfrak{p} \mathcal{N} \mathfrak{p} \)
4. A \underline{filtration} \( \text{Fil}^i D , \text{ie, } \mathbb{Z} \)

\( \text{st} \)

\( \text{Fil}^i D = \text{Fil}^i (D) \)

\( \cup \text{Fil}^i D = D \quad \cap \text{Fil}^i D = 0 \)
Easy to check that if \( f : G_{\mathfrak{f}} \to \text{Act}_{\mathfrak{Q}}(V) \) is a non-rep'\( ^\prime \),

then \( \dim_{\mathfrak{Q}}(D_{\mathfrak{f}}(p)) \leq \dim_{\mathfrak{Q}}(V) \). If equality holds, \( V \) is said to be semi-stable.

Further, \( D_{\mathfrak{f}}(p) \) is a filtered \((\mathfrak{Q}, N)\)-module.

Weakly admissible

If \( D \) is a filtered \((\mathfrak{Q}, N)\)-module

\[
\text{Define } t_H(D) = \sum_{i \in \mathbb{Z}} \text{dim}(\frac{\text{Til}^i(D)}{\text{Til}^{i+1}(D)}) \cdot i
\]

& \( t_N(D) = \sum_{\alpha \in W} \text{dim}(D_{\alpha}) \cdot \alpha \)

slope of generalized eigenop.

\( t_N(D) = \sum_{p \in W} v(p) \times \text{v}(x) \)

roots of char poly of \( p \)

\( v(p) = 1 \).

\( D \) is weakly admissible means

\( 1. t_H(D) = t_H(D) \)

\( 2. t_H(D') \leq t_H(D) \) for all \((\mathfrak{Q}, N)\)-stable subobject of \( D \).

\( \text{i.e. } D' \subseteq D \) s.t.

\( 1. q(b') \subseteq D' \)

\( 2. N D' \subseteq D' \)

\( 3. \text{Til}^i D' = \text{Til}^i (D \cap D') \).

Trostam checked that if \( V \) was semi-stable

then \( D_{\mathfrak{f}}(V) \) was weakly admissible.

\& the functor \((\text{semi-stable} \quad \text{adm. rep'}\( ^\prime \)) \to (\text{w. fitting} \quad \text{\((\mathfrak{Q}, N)\)-module}) \) was fully faithful.

Much more recently, Trostam & Colmez, showed

Upshot. if \( V \) is semi-stable, we can recover \( V \) from \( D_{\mathfrak{f}}(V) \)

\& we can "list" all the possibilities for \( D_{\mathfrak{f}}(V) \).
if $P$ is a modular form of level $N$ (not $N'$) and $P_{N_p}$ has trivial $N_p$-multiplicities, then $P_{p} \mid P_{N}$ is semi-stable.