Mar 9, 2006. Thursday. 1:00pm - 2:20pm
Kevin Buzzard (8th)

101 ≤ 1 → nonconvergent series?

Recall I talked about

\[ W \]
\[ W^+ = \text{half of them} \]
\[ W^0 = \text{one of them} \]

1-adic \( \mathbb{F}_p \)-extension \( W^+ \)

Eisenstein family lives over \( W^0 \)

\( V_k \in W^0 \) there's a power series

\[ \exp V_k = 1 + \sum_{n=1}^{\infty} \frac{V_k^n}{n!} \in \overline{Q}_p \]

& \( \mathcal{O}_{E_\infty} \) \( V_n \geq 1 \)

\( \mathcal{E}_k \in \mathcal{O}_{E_\infty} \) \& \( \mathcal{E}_k = 1 \in \overline{\mathbb{F}_p} \)

Integers of \( \mathcal{O}_p \)
Recall a $q$-exp $F \in \mathcal{Q}_g^R$ is an overconvergent modular form if $\omega$ be $\omega_0$, if $F/E_k$ is the $q$-exp $^g$ of an overconvergent modular form.

Indeed, if $G$ with $\omega \Rightarrow F \in \omega$, $k \in \omega$

If $F$ is overconvergent at $\omega$, then $T$ is a Hecke operator

Then $T \omega$ is overconvergent.

Here's one proof.

Let $\omega$ be any non-constant.

$U : R \mathcal{Q}_g^R \rightarrow R \mathcal{Q}_g^R$, $U(\Sigma_{\omega_0} \omega_0) = \Sigma_{\omega_0} \omega_0$

$V : R \mathcal{Q}_g^R \rightarrow R \mathcal{Q}_g^R$, $V(\Sigma_{\omega_0} \omega_0) = \Sigma_{\omega_0} \omega_0^+$, $U \circ V = \omega$, $(V \circ U) \circ \omega$

and more generally

$U(F \mathcal{Q}(G)) = \mathcal{Q} \mathcal{U}(F)$, $T \mathcal{G} \in \mathcal{Q}_g^R$

Recall there's a Hecke operator $U$ on overconvergent modular forms.

Let us explain why. If $F$ is overconvergent

Then $U(F)$ is too.

It's a theorem of Coleman that if $V_k$ denote $V(E_k)$

Then $E_k/V_k$ is the $q$-expansion of an overconvergent modular form.

(Q) How far does it overconverge? I

Using this, then, we easily deduce that $U$ preserves cut at

Say $F$ is overconvergent at $\omega$

Then $F = E_k G$, $G$ overconvergent at $\omega$

$U(F) = U(G, E_k)$

$= U(G, E_k)$, $\frac{V_k}{V_k}$

$= U(G, \frac{E_k}{V_k}) = E_k \times U(\frac{G \circ E_k}{V_k})$

$\frac{U(F)}{E_k}$ is overconvergent at $\omega$. $U(F)$ is overconvergent at $\omega$

This argument proves that $U$ is $q$-exp $^g$ of form with

as long as $r$ is small.

Thus, for $n$-overconvergent $\omega_0^g$ means $\omega_0$ on ordinary locus.
because if $G$ is $\varepsilon$-overconvergent, then $G \times E_k \rightarrow \frac{V_k}{pK}$

$U$ of it is $p\varepsilon$-overconvergent.

Now restrict back to $\varepsilon$-overconvergent

Argument shows this

If $\varepsilon$ is small, define $\varepsilon$-overconv. set $k$ forms

$:= \varphi \exp' \left( \chi \right)$ s.t. $\text{F}_k/E_k \nabla \varepsilon$-overconvergent.

$U$ is a d.c. map on $\varepsilon$-overconvergent map $U \rightarrow \varepsilon$-overconvergent

$\text{opt} \downarrow \text{ree} \left( \text{opt} \right)$

$U$-action on $\varepsilon$-overconvergent

Some set $k$ have a Char. power series $P_k(T) = \text{det}(1 - TU)$

Just as in w.t.o. this QPS is indep of $\varepsilon > 0$.

Here's an idea then:

For any $\delta \in W^o$, plot the zeroes of $P_k(T)$

in $\mathbb{A}^1 / \mathbb{F}_p$ (wonder space)

As $k$ varies, you get a subset of $W^o \times \mathbb{A}^1$

$p(k), E_0 = 1 \& \delta \times k$ is close to $0$,

$E_k = 1 \mod \text{big power of } p$

This picture is called the spectral curve associated to $\delta$. 
How does one construct it properly?

Remember the lecture when I did Serre’s Endomorphisms’ paper?

I defined a CPS for a cpt operator on an ONtable pre-Ronch space.

We want to generalize this to ONtable Banach modules over a complete ring of some kind.

E.g., let $R$ be the ring $\mathbb{Q}[T]$. Norm $\|R \| = \max |a_i|$

Can define a Banach module over $R$ as a complete normed $R$-module $M + axioms \| m \| \leq 1 \| m \|$, etc.

Key example: an ONtable one.

Pick a set $I$. E.g., $I = \{1, 2, 3, \ldots\}$

Set $M = \text{fns } f : I \rightarrow R \\text{ s.t. } f(i) \rightarrow 0 \text{ as } i \rightarrow \infty$

One defines $\text{cts}$ & $\text{cpt}$ operators on such things.

Finite rank: $\text{Fin } f \leq \text{fns } \text{ gen. } R$-module

Cpt: limit of finite rank

Cpt ops have a CPS of some $d \in R[[T]]$

Idea: if $D \subset W^0$ is a small closed disk, let’s define the

$\text{O}(D)$-module of $\varepsilon$-overconvergent forms on $D$

to be the $\text{fns}$ on $D \times X(N)$.

Remark: I am thinking about an overconvergent at $\kappa$

form as being equal to an overconverged form on $X(N)_{\text{ord}}$.

Define the $\varepsilon$-expansion of such an object $m$ being the $\text{cuntz}$

$\varepsilon$-expn on $(\text{O}(D) [\varepsilon])$

$\times E_D$ where $E_D$ is $\varepsilon$-expansion of Eisenstein“Heart” over $D$. 
\[ E_k = 1 + \sum_{n \geq 1} a_{n,k} g^n \]

where \( A_n \) is a \( \mathbb{F}_q \)-valued Eisenstein family at once, as being an element of \( \mathcal{O}(W^0) \mathbb{F}_q \).

\[ 1 + \sum_{n \geq 1} A_n g^n \]

In fact one can check that \( W \) is the usual parameter of \( W^0 \) space.

\[ k(1+\ell) = w+1 \]

\[ \text{if } s \neq 0 \]

then \( E \in \mathbb{Z}_p \mathbb{F}_q \).

This is a "computable object."

In fact \( \frac{2}{s} \in \mathbb{W}_p \mathbb{F}_q \) (pole of \( s \) at \( w_0 \)).

\[ E \in 1 + w g \mathbb{Z}_p \mathbb{F}_q \mathbb{W}_p \mathbb{F}_q \]

Define \( V = \mathcal{V}(\mathbb{F}_q) = \mathbb{E}(g) \in 1 + w \mathbb{Z}_p \mathbb{W}_p \mathbb{F}_q \)

\[ E/V \in 1 + w \mathbb{Z}_p \mathbb{W}_p \mathbb{F}_q \]

one can now specialize to \( w = w_0 \in W \) weak

& recover \( E_k/V_k \).

Explicit analysis of \( 2 \)-adic spectral curve near boundary of \( W_0 \).

Exciting new parameter!

\[ E_2 = \text{classical } 2 \text{-level } \mathcal{E}_2 \text{ Eisenstein} \]

\[ \frac{1}{1 + 2 A (g + g^2 + \ldots)} \]
\[ V_2 = V(E_2) \] classical w/ 2 levels

\[ \frac{E_2}{V_2} = 1 + 2t + \ldots \] meromorphic on \( X_0(4) \)

Set \( y = \frac{E_2}{V_2} - 1 \) = \[ 9 - 20q + 462q^5 + \ldots \in \mathbb{Z}[q] \]

\[ y : X_0(4) \rightarrow P^1 \]

Recall \( f = \frac{\Delta(q)}{\Delta(q^5)} = X_0(2) \rightarrow P^1 \) & one checks that

\[ f = \frac{y + 8qy^2}{1 - 8qy^2} \]

In particular, the natural map \( X_0(4) \rightarrow X_0(2) \) induces an isomorphism between regions \( |q| \leq 1 \) & \( |q^2| \leq 1 \).

So more generally between \( |q| \leq d \) & \( |q^2| \leq d \) & \( d \leq 8 \)

\[ X_0(2) \]

ord. locus
\[ \begin{array}{c|c|c}
|q| & |q^2| & d \\
\hline
1 & 1 & 1 \\
1 & d & d \\
1 & 8 & 8 \\
\end{array} \]

due: \( q \) & \( y \) are parameters

Now we use powers of \( y \) instead of \( q \)

What is the matrix of \( U \) on overconvergent at \( \mathfrak{m}_2 \) some

sort basis \( V_k, V_k(qy), V_k(qy)^2, \ldots \in F \sqrt{2} \\

One can answer this question \( \iff \) one knows how to write

\[ E_k/V_k \] as a power series in \( y \)

Here's what I know

If \( k \in W^0 \) & the parameter \( W = w(s) = h(s) - 1 \) satisfies \( |w| \leq \frac{1}{8} \)

then Kilford & I showed that \( E_k/V_k \in \mathcal{O}_2 [q, qy] \)

Next time I'll show you why.
Next time I'll show you why.

In fact we really prove that

\[ E/V \in \mathbb{Z}_5 \{ w, y \} \] is also in \( \mathbb{Z}_5 \{ \frac{w}{8}, y \} \)

\[ |w| < \frac{1}{5}, |y| < \frac{1}{8} \]
\[ |w| = \frac{1}{5}, |y| = \frac{1}{8} \]

\[ E/V \in \mathbb{Z}_5 \{ w, y \} \cap \mathbb{Z}_5 \{ \frac{w}{8}, y \} \]

\[ = \mathbb{Z}_5 \{ w, w', y, y' \} \]

**Consequence:** If \( w = w_0, w_0 \in C_2 \), \( |w| > \frac{1}{8} \), \( w \in W \)

then \( E/k \in C_2 \{ w, y \} \)

What happens is that \( E/k \) being very overconvergent in center of cut space.

\[ \Rightarrow \text{a little overconvergent at boundary} \]

Moreover, if \( E/k = q_k(w_0, y) \cdot q_k \in C_2 \{ x \} \)

then for \( |w| > \frac{1}{8} \), \( q_k \in C_2 \{ x \} \) is independent of \( k \)

as one checks easily that \( q_k = \sum_{n=0}^{\infty} C_n \cdot x^n \)

\[ \text{if } E/V = \sum_{ij} q_{ij} w_i y_j \]

One can even compute \( q_k \) by choosing one \( k \in W \)

near boundary & hashing it out

e.g. choose \( k: \mathbb{Z}_3^x \rightarrow \mathbb{C}_2^x \)

\[ \chi(k) = \{ x \mid x = 1 \mod 4 \} \]
\[ \cup \{ x \mid x = 3 \mod 4 \} \]

\( k \leftrightarrow \text{classical point } (\chi, \bar{k}). \) \( k = 1 \)

\( W = k(5)-1 = 4. \) \( |w| = \frac{4}{5} > \frac{1}{8} \)

One checks that \( E_k = \sum_{a, b \in \mathbb{Z}_3^x} q^{a+b} = 1 + a q + \cdots \)
$V_k = E_k(g^2)$ level $\theta$.

$T_k = E_k/V_k$ is a modular form of level $\theta$.

One wants this as a power series in $g$.

One checks that $T_k = (1+8g)k + (1+8g)^{-1}$ can solve $\sum g_k = \sum_{n=0}^{\infty} x^{a_k} - 1$.

Just as in $\theta$, one gets one teeth. Can compute $U(\mathbb{C} g^2)^n$ as a power series in $g$

Then $U(V_k(\mathbb{C} g^2)) = E_k(\mathbb{C} g^2)^n = E_k, V_k, U(\mathbb{C} g^2)$

Now get a formula for matrix entries $g_k(\mathbb{C} g^2)$

For some entries, all we know is a lower bound on valuations.

For some entries, we know what valuation is.

Big matrix representing $U$ must look like

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

One checks easily that one can throw away all rows & corresponding columns & not change CPS.

The new matrix has the property that $W_i$ divides i-th row & furthermore, if you divide i-th row by $W_i$, the resulting matrix has the property that det $S_i$ is the lowest

and $W_i$ is always a unit.

$\Rightarrow$ slopes of CPS are $1, 1(w), 2v(w), 3v(w)$.

Pictures of 2-edge spectral curve $\ldots$