Recall notation:

\[ \mathcal{O} = \frac{\Delta(q^d)}{\Delta(q)} \]  

\( \Phi: X_0(2) \to \mathbb{P}^1 \)

\( \Phi: X_0(2) \to \mathbb{P}^1 \)

\( \frac{1}{3} = \frac{\Phi}{(1+256 \Phi^2)} \)

\( \frac{1}{3} = \Phi + \cdots \in \mathbb{Z}[\frac{1}{3}] \)

In particular, \( \frac{1}{3} = \Phi + \cdots \in \mathbb{Z}[\frac{1}{3}] \)
Barten: \( \frac{2^g}{3} = 2^{g-1} \cdot e \cdot \mathbb{Z}_2 \{2^g \cdot 3 \} \)

\[ 2^{g-2} = \frac{2^g}{3} + e \cdot \mathbb{Z}_2 \{2^g \cdot 3 \} \]

- even if \( |g| \) is a little less than \( 1 \)
- there's a canonical \( g \)-associated \( \delta \)
- i.e. a section of the "forget" map.

So picture is

\[ X_0(1) \]

Recall a 2-adic modular form is a (2-adic) holomorphic form on \( X_0(1) \) such that extends a little into the missing disc

\( |\delta| > 1 \) = closed disc \( |\delta| \leq 1 \) = closed disc \( |\delta| < 1 \)

i.e. an element of \( \mathbb{Q}_\ell \langle \delta \rangle \)

An overconvergent 2-adic modular form is a form on \( X_0(1) \) such that extends a little into the missing disc.

e.g. choose \( \delta \), \( 1 < |\delta| \)

Consider the subgroup

\( \mathbb{Q}_\ell \{2^g \} \)

- any element \( f \) is overconvergent
- the bigger \( \delta \) is, the more overconvergent you are

**Hecke ops**

Classical modular forms come with Hecke operators

\( T_2, T_3, T_5, \ldots \)

What are Hecke operators?

If a modular function is a "function on elliptic curves" (Katz)

\[ E \rightarrow f(E) \]

then \( T_q f \) (at prime) is

\[ \frac{1}{q} \sum_{c \in E} f(E/c) \]
In our $p$-adic setting, a $p$-adic modular function is a form on ordinary elliptic curves. It's well-known that anything *almost* to ordinary is ordinary. So it makes sense! 

$$L \supset \mathbb{C} \stackrel{P(E/C)}{\rightarrow} \mathbb{C}$$

Get a bunch of continuous endomorphism of the space of $p$-adic modular functions. So they all commute.

A nice thing to have, though would be one compact operator.

If $\varphi : V \rightarrow V$ is opt.

then we can start pulling off finite-dim subsp. of $V$ (generalized eigenspaces for $\varphi$. non-zero eigenvalue.)

So they will be stable under all the other Hecke operators.

I don't know any natural such things though.

However, if we consider overconvergent forms, one does appear.

Recall from last time.

If $E/O$, that supersingular but not too supersingular reduction, then amongst the 3 reps of order 2, one sticks out, with respect to valuations of coordinates.

**Formal groups:**

$$x = \frac{y}{g^2}$$

Parameter for the formal gp of elliptic curve.

$g^2$ = cubic of $x$.

What I did last time was to show that the 3 points of order 2 in the formal gp of the curve were in 2 classes.

duo to had one norm the third one (canonicale) had a different norm.

Formal gp = $O_2 \times \mathbb{Z}/2 \mathbb{Z}$

$[2]z = \text{power series in } z$
Zeros of power series

\[ \text{pts of order dividing } 2 \text{ in formal } \mathbb{G}_p \]

\[ \mathbb{G}_p \]

\[ \mathbb{E}/\mathbb{G}_p \]

\[ \pi : \mathbb{E}/\mathbb{G}_p \rightarrow \mathbb{E}/\overline{\mathbb{G}_p} \]

\[ (x \mod 2) \rightarrow \mathbb{E}/\overline{\mathbb{G}_p} \]

In ordinary case, amongst the 3 pts of order 2, only one is in the disc.

In supersingular case, all 3 pts of order 2 are in disc.

Ordinary case

\[ [2] Z = 2Z + \text{const } Z^2 + \ldots \]

& (Newton polygon) \(
\begin{array}{c}
\text{const} \\
\text{is a 2-adic unit}
\end{array}
\) \( Z = -\frac{a}{2} \) is close to another root.

In non-ordinary case,

\[ [2] Z = 2Z + a^2 Z^2 + b Z^3 + c Z^4 + \ldots \]

\( a \) is now a unit.

\( a \) is not a unit.

But \( 2 | b \) because \( \text{mod } 2 \mathbb{G}_p \).

\[ [2] Z \text{ is a lift of } Z^2. \]

Newton polygon when \( |a| < 1 \) but only just.

\[ \text{If } v(a) < \frac{2}{3}, \text{ one can spot a canonical root.} \]

All done for general \( p \) in Katz's paper.

\( p > 2 \) no harder than \( p = 2 \).
\(Q = 0 \quad \text{ord. case} \)

\(Q > 0 \quad \text{but } \frac{Q}{p} < \frac{p}{p+1} \)

get \(p-1\) canonical sel's & \([p]Z = 0 \quad \& \quad \text{these are the canonical subgp.} \)

**Upshot:** the canonical subgp of an ell curve \(E/K\) depends only on the formal gp associated to \(E \times_0 (2) \)

\[
\begin{array}{c}
|F| \leq 1 \\
(E, C) \quad \text{C canonical} \\
C \text{ will exist when } \langle a \rangle \text{ is small}
\end{array}
\]

**Miracle:** if \(d\) is prime, \(d \neq 2\), then \(E\) & \(E/D\) have isomorphic formal groups!

\[
P \subset E \text{ a subgp of order } d
\]

\[
E \xrightarrow{\varphi} E/D \xrightarrow{\varphi^p} E
\]

End (formal \(q^t\)) \(\cong \Z_p \times \Z \)

\[
\text{& } d \in \Z^x
\]

So the Hecke operators

\[
T_d \& \varphi \text{ don't change } |F| \text{ as long as } 1 < |F| < \text{ small thing}
\]

**Conclusion:**

\[
T_d \text{ acts on } \Q(2^t \varphi) \text{ for } 1 \leq t \leq 7 \text{ as well if } d \neq 2.
\]

\(
\bullet \) Unfortunately, \(T_2\) doesn't preserve \(\Q(2^t \varphi)\) if \(1 \leq t \leq 9\)

\(T_2\) "makes things coarse"
Can we see what's happening?

\[ X_0(y) \]

Say we know \( j(E) \) & it has norm 1-2.

Let \( C \) be one of the subq's of \( E \) of order 2.

What is \( j(E/C) \)?

Answer is given by the classical modular polynomial

\[ \Phi_2(X, Y) \]

a polynomial deg 3 in each variable s.t \( \Phi_2(j(\omega), j(\omega')) = 0 \)

\[ E = \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/4 \]

\( \mathbb{Z}/4 \) contains

\[ E/\mathbb{Q} = \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/2 \]

Recall

\[ \Phi_2(X, Y) = X^3 + (-Y^2 + 1488Y - 162000)X^2 \]

\[ + (1488Y^2 + 7703375Y + 8743800000)X \]

\[ + (y^3 - 162000Y^2 + 8743800000Y - 1574640000000) \]

Check \( \text{sub. in } Y = 2 \)-adic unit

All 3 roots for \( X \) should be units \( \omega \)

Now \( \text{sub. in } Y = y \in \mathbb{Q}_2 \)

\[ 1 - \omega = |y| < 1 \]

Hope: all 3 roots for \( X \) have \( |y| < 1 \) & \( T_2 \) should exist
**Newton polygon**

![Newton polygon diagram]

**Conclusion:**
- If \( v(j(E)) = 0 \), \( c = 8 \), then \( v(j(E/c)) \) is either \( 2c \) (once)
- \( E/c \) s.s. reduction, \( v(j(E)) = 8 > 0 \) \( \frac{a+2c}{2} \) (twice)

3 subgps order 2, of which 1 is different
- 3 answers for \( j(E/c) \), one of which is different.

**Fact:** Obvious guess is correct! The canonical subgp gives rise to a totally different \( j \)

So in fact, \( T_2 \) breaks up into 2 Hecke operators

\[
(T_2 \Phi)(E) = \frac{1}{2}(\Phi(E/c_1) + \Phi(E/c_2) + \Phi(E/c_3))
\]

\( \Phi \) if \( E \) is ordinary or not \( \frac{a+2c}{2} \) supersingular
then one of the \( c_s \), say \( c_1 \) is canonical

Define \( (V\Phi)(E) = \frac{1}{2} \Phi(E/c_1) \)

\& \( (U\Phi)(E) = \frac{1}{2} (\Phi(E/c_2) + \Phi(E/c_3)) \)

\( U \) involves elliptic curves s.t. valu of \( j \)-invariant
has gone down.

\( U \) (but not \( V \)) will induce an endomorphism of \( \mathbb{Q}_2(\zeta_2, \Phi) \)

(\& all this works for general \( N, p \))

(Play around with power series \( \{p j \mathbb{Z}_p \} \))

In fact more is true \( V \) of \( j \)-invariant was being divided by \( 2 \)
when we applied \( U \).
Pictorially,

\[ |s| < 1 \]

So we in fact see that \( U \) is a cont. map

\[ \mathbb{Q}_p \langle 2^t f \rangle \rightarrow \mathbb{Q}_p \langle 2^{2t} f \rangle \rightarrow \mathbb{Q}_p \langle 2^{3t} f \rangle \]

for \( t \in \{1, 2, 3\} \)

More generally, \( U \) is a continous map from "r-overconvergent \( p \-\)adic modular forms" to "i-r-overconvergent \( p \-\)adic modular forms".

\( U " \) increases over convergence."

Consequence:

The induced map \( U : \mathbb{Q}_p \langle 2^t f \rangle \rightarrow \mathbb{Q}_p \langle 2^{2t} f \rangle \) (\( t \leq 3 \)) is cont.

\( \textbf{Pr.} \) \( U \) is composition of a cont. map & the inclusion.

\[ \mathbb{Q}_p \langle 2^t f \rangle \rightarrow \mathbb{Q}_p \langle 2^{2t} f \rangle \& \text{ this is cont.} \]

\[ \text{cont. composites = cont.} \]

\( \mathbb{Q}_p \langle T \rangle \) has an ON basis

\[ 1, T, T^2, T^3, \ldots \]

\( \text{w.r.t. the obvious basis } 2^t \times 2^t f \)

the inclusion \( \mathbb{Q}_p \langle 2^t f \rangle \rightarrow \mathbb{Q}_p \langle 2^{2t} f \rangle \)

has matrix \( \begin{pmatrix} 1 & 2^t \\ 2^t & 2^{2t} \end{pmatrix} \)

Bond entries.

All \( \leq 2 \) under heat line.
Remark
\[ U = U_2 \text{ on } X_0(2), \quad U_2 f(\ell, \mathcal{O}) = \frac{1}{2} \sum_{D|\ell} f(D, \mathcal{O}) \]

Effect on \( q \)-expansions

An elementary computation with Tate curves gives that
if \( f = \sum a_n q^n \) is a \( \mathbb{Q} \)-adic modular form, then \( U f \)
has \( q \)-expansion \( \sum a_n q^n \).

More generally, for any \( f \),
\[ U(\sum a_n q^n) = \sum a_{np} q^n \]

\& \quad V(\sum a_n q^n) = \prod (1 - q^n) \sum a_n q^n.

To make our life easier, let's redefine \( V \)
so \[ V(\sum a_n q^n) = \sum a_n q^n. \]

In general, \[ V(\sum a_n q^n) = \sum a_n q^n \]
is an endomorphism of space of \( p \)-adic modular forms (not over \( \mathbb{Q} \)).