Hecke ops & Diamond ops.
are really quite general things
big space \( M \) + action of big group \( \Gamma \)
\( \Gamma = \Delta \), \( \mathbb{H} \) is Riemann points
If \( \Pi_1 \subset \Pi_2 \) then induced \( \Pi_1 \leq \Pi_2 \)
If \( \gamma \in \Delta \), then \( \gamma \rightarrow g, \gamma \) induces a map
Finally if \( \Pi_1 \triangleleft \Pi_2 \) finite index
Then \( \Pi_2 = \bigcup_{\ell \in \Lambda} \Pi_1, \ell \Pi_1 \)
then \( \gamma \rightarrow \sum \chi_\ell, \gamma \) is a map \( \Pi_{\Pi_1} \rightarrow \Pi_{\Pi_2} \)

Hecke operators: explicit instances of these ideas

Example of how Hecke ops act on \( g \)-expansions

If \( f \in M_k(\Gamma_1(N)) \)
\( \forall \gamma \in \Delta \), \( \gamma f = \chi(\gamma) f \)
for some Dirichlet char.
then If I know \( g \)-exp of \( f \), I can compute \( g \)-exp of \( T_\gamma f \)
\( f = \sum a_n \gamma^n \)
If \( \gamma \mid N \), \( T_\gamma f = \sum a_n \gamma^n \)
\( \rho_{\gamma}(N, f) \in \mathbb{H} \)
& is \( \gamma \mid N \), \( T_\gamma f = \sum a_n \gamma^n + \sum \chi_\ell \gamma(\ell) \sum d_\ell \gamma^n \chi_\ell(\gamma) \sum d_\ell \gamma^n \)

One key consequence: \( f = \sum a_n \gamma^n \) is an eigenform for all the \( T_\gamma \), then one deduces that \( a_\gamma \) is the eigenvalue of \( T_\gamma \)

In fact one can use this to explain
why many common modular form have \( g \)-expansions in \( \mathbb{Q}[[g]] \).

General theorem: eigenvalues of \( T_\gamma \) are algebraic numbers.

So \( \mathbb{H} \), \( \Xi(N) \) etc are compact Riemann surface.

Exercise: \( \Pi_1(M) \) naturally bijects with \( \text{pair } (E, P) \) elliptic curve/pt of order \( N \)
Dictionary: \( \mathbb{Z} \otimes \mathbb{Z} \rightarrow (\mathbb{C}/\mathbb{Z}, \mathbb{N}) \)

Amazing fact: this simple idea can be translated into alg. geo.

\[ X_\ell(2) \cong \text{concrete description} \]

**Facts:** If \( \Gamma \leq \text{SL}_2(\mathbb{Z}) \) is a congruence subgp & \( \Gamma \) has no elliptic elements, then there is an alg. curve \( Y(\Gamma) \) over \( \mathbb{Q} \) parametrizing elliptic curves (over \( \mathbb{Q} \)-schemed with a level \( \Gamma \) structure)

eg. if \( \Gamma = \Gamma_1(N), \ N \geq 4 \)

then "level \( \Gamma \) structure" can mean "point of order \( N \)" on embedding of \( Y_1(N) \)

More precisely, there's an alg. curve \( Y_1(N)/\mathbb{Q} \).

For an elliptic curve \( E \) equipped with a point \( p: Y_1(N) \rightarrow E \) of order \( N \)

\[ \mathbf{S} \rightarrow Y_1(N) \] is any scheme over \( \text{Spec} \mathbb{Q} \)

\( Y_1(N)(K) \) bijects naturally with \( 130 \) classes of pairs \( (\lambda, Q) \)

\[ \text{A d\text-scrt}(K) \]

\( Q \in \text{E}(K) \) pt of order \( N \)

There are simple construction of \( Y_1(N) \) & fancy ones too.

Similarly \( Y(N) \) for \( N \geq 3 \) is an alg. curve/\( \mathbb{Q} \)

- parametrizing pairs \( (E, q: \mathbb{Z}_N x \mathbb{Z}_N \cong E[1]) \)
- Algebro-geometric \( Y(N) \) & \( X(N) \) & \( \mathbb{Q} \)-line preserving natural pairing

have associated Riemann Surfaces is to be called \( Y(N), Y_1(N) \).
What about $Y(N)$. N small.

& What about $Y_0(N)$, trying to parameterize pairs $(E, C)$ of cycle sp to $O(N)$ of ord $N$.

These things also exist in alg. geo. because in alg geo, you can sometimes forget cat by a finite sp. & you can fine.

General construction of $Y_0(N)$: choose an prime $p$, $p+N$.

The modules problem representing $E$, $C$ full level $p$-str.

is representable by a curve $\text{Spec } \mathcal{A}$.

& the curve is $\text{Spec } \mathcal{A}$.

& it has a natural action of $\mathbb{Z}/p\mathbb{Z}$.

Invariants $\mathcal{A}$ take $\text{Spec } \mathcal{A}$.

Similar trick does $Y_1(N)$, $N \neq q$.

These curves $Y_0(N)$ & $Y_1(N)$ aren't quite the solutions to

natural mod problems involving elliptic curves.

but there's still a canonical bijection of sets

$Y_0(N)(K) = \text{two classes of pairs } E, C$.

If $N \neq q$, then there's a sheaf $\omega$ on $Y_1(N)$

whose analytification is $\omega$ of last time.

$E$

$\downarrow^\pi$

$Y_1(N)$

$\omega$ is an invertible sheaf on $Y_1(N)$

fiber of $\omega$ at $P$.

However, "quotient finite" doesn't exist on $\omega$.

& $Y_0(N)/\mathbb{Q}$ etc don't have a natural $\omega$.

$\zeta$ although they may have a natural power of $\omega$, eg. $Y_1(N)$ has $\omega$ on it.
Curves $X_0(N) \times X_1(N)$ have natural compatibilities $X_0(N) \times X_1(N)$

One last remark:

If $X_1(N)$ is the modular curve $/ \mathbb{Q}$

$H^0(X_1(N), \omega^{\otimes k})$ is a $\mathbb{Q}$-lattice in the $M_k(\Gamma_1(N))$ of $\mathbb{Q}$-linear form $\mathbb{Q}$-v. of

Custom fact: If $f \in M_k(\Gamma_1(N))$, then $f \in M_k(\Gamma_1(N), \mathbb{Q})$

[$*$ - expansion in $\mathbb{Q}(i)$]*

Remark: For this to be really true,

I should define $X_1(N)$ as parametrizing $(E, \beta)$, $\beta : (c, \infty, E)$

Note: $p$-adic properties of modular schemes is modular forms.

In particular: $C_k = \frac{x(z)}{2} + \sum_{f \in \mathfrak{S}} f^k \in M_k(\Gamma_1(N), \mathbb{Q})$

And... strongly to this course.

1. $p$-adic theory - Eigencurves. In particular, writing eigencurve up

Modular form have yet good $p$-adic analytic properties.

Example: one can check that for $p$: prime,

If $S = \{ k \in \mathbb{Z} : k \geq 2 \text{ and } k \equiv 0 \text{ mod } p-1 \}$

$p$ odd

$\sum_{k \in S} \sum_{n \geq 1} \frac{1}{k-1} (n^k - 1)^2 \frac{1}{n!} \phi^k \{ \frac{1}{n} \}$

$p$ even

$E_k \equiv 1 \pmod{p}$
1. is the $q$-expansion of a modular form at $k$. level $p$.

2. varies $p$-adically continuously with $k \in \mathbb{S}$

   - one can check (Colesam 2.12 & remarks after lemma 9.7)
     Washington "holomorphic field"

     that $q$-expansion of $E_k$ is in $\mathbb{Z}_p[[q]]$

     $x$ is $k \equiv k' \mod p^n$ then $E_k \equiv E_{k'} \mod p^n$

Hence Eisenstein series seem to move $p$-adically continuously.

Does $M_k(\pi(N), \mathbb{Q})$ move $p$-adically continuously?

No, is $k \equiv k' \mod p^n$

but $k \not\equiv k' \mod p^n$

then $M_k \not\equiv M_{k'}$.

Coleman saw how to expand $M_k(\pi(N))$ so that was $p$-adically $k$-adically close.

In fact,

$S \subset W = \text{open } p$-adic disc.

& Coleman defined "overconvergent modular forms of weight $k$" for any $k \in W$.

3. p-adic Langlands for $Gln$.

Let $f$ be $\mathcal{S}_k(\pi(N))$ be an eigenform.

$f = \sum_{n=0}^\infty a_n q^n$

coeff. of $q$ generate a number field $E$.

If $l$ is a prime of $E$, ALL
then Deligne (for $k \geq 3$)

{Eichler Shimura (for $k=2$)} associates to $f$ a Galois repsn

{Deligne-Serre (for $k=1$)}

\[ \rho_k : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(E_{\mathbb{Q}}) \text{ satisfying certain properties.} \]
e.g. if $p$ is prime, $p + N$, then $F_p$ is unram. at $p$

So $F_p(\text{Frob}_p)$ has char. poly.

$X^2 - a_p X + p^{k+1}$

where $Frob_p$ is an element of $T_f$

$\mathbb{F}_p$ is an eigenspace of $T_f$

& $F_p$ is called an irreducible

$F_p$ has Hodge-Tate weights $0$ & $k-1$.

Q) What is $F_p|_{\mathbb{F}_p}$ for the other primes $p$?

* Add: Here's the answer for $p = 2$: Associated to $f$ is an infinite dimensional repn $\Pi_p$ of $GL_2(\mathbb{A}_\infty)$.

This is not a mystery. Consider $\mathbb{F}_p \otimes \bigotimes Q

& one checks $\Pi_p = \bigotimes_{\mathbb{F}_p} \Pi_p$

e.g. if $p + N$, $\Pi_p$ is "unramified principal series"

& is determined by $Q_\mathfrak{p} \otimes \mathbb{F}_p \otimes \mathbb{F}_p$

Local Langlands conj for $GL_2$ ← a theorem

Associates to $\Pi_p$ as above.

a 2-dim repn of the Weil-Deligne gp of $\mathbb{Q}_p$.

& then an elementary construction gives a 2-dim rep. $\left( \mathbb{F}_p, p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to GL_2(\overline{\mathbb{Q}_p}) \right)$

"Local-Global compatibilities" (thin of Carayol)

$\mathbb{Q}_p$ is the restriction of $\mathbb{Q}_p$ to $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

In particular, our previous statement about $F_p|_{\mathbb{Q}_p}$ for $p + N$

is a conseqt. of this.
Prf: One can almost prove Local Langlands for GL_2 by writing down both sides.

What about $p = 2$?

$\mathbb{F}_p \rightarrow \mathbb{F}_p^* \rightarrow \mathbb{P}_p \rightarrow \mathbb{C}_p^* \rightarrow \mathbb{C}_p^*$

$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{C}_p^*$

These are lots of these.

$\mathbb{P}_p$ is not that many of these.

He will write down a lemma.

Q1) Can we recover $\mathbb{P}_p$ from $\mathbb{P}_p^*$?

Yes (T. Seitz)

Q2) Can we go other way? Answer must be "no".

So what can we do?

1) Use global nature of $\mathbb{P}$ to put a little extra stuff on $\mathbb{P}_p$.

2) Prove $p$-adic Local Langlands for GL_2.